

SHRINKING AND GLUING UNDER LOWER CURVATURE BOUND

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ABSTRACT. We show that a distance non-increasing onto map $f : \amalg Z_\alpha \rightarrow X$ between Alexandrov spaces preserves volume if and only if it preserves the length of path. This implies the converse Petrunin's Gluing Theorem: if the gluing of two Alexandrov spaces is an Alexandrov space, then the gluing is along the boundary and via an isometry. To prove the main result, we develop a new technique to approximate the distance in Alexandrov space subject to dimension comparison.

INTRODUCTION

Recall that Alexandrov space is a length metric space with curvature bounded from below in the sense of triangle comparison (Toponogov's Theorem) ([BGP]). Our work was motivated by the gluing theorems first proved by Perel'man and then generalized by Petrunin. Roughly speaking, by gluing we mean to identify points in several spaces and equipped the new space with the induced length metric. In 1991, Perel'man proved the following Doubling Theorem [Per] and it has been widely used in Alexandrov geometry.

Theorem 0.1 (Perel'man, [Per]). *The doubling (the gluing of two copies along their boundaries via an identity map) of an Alexandrov space with boundary is an Alexandrov space with the same lower curvature bound.*

In 1997, Petrunin generalized the above theorem to the gluing of two different Alexandrov spaces.

Theorem 0.2 (Petrunin, [Pe1]). *The gluing of two Alexandrov spaces via an isometry between their boundaries produces an Alexandrov space with the same lower curvature bound.*

The converse theorem was asked by Petrunin: assuming that an Alexandrov space is glued from two Alexandrov spaces via an identification $x \sim \phi(x)$, where $\phi : \partial X \rightarrow \partial Y$ is a one-to-one map with $\phi(\partial X) = \partial Y$, does ϕ have to be an isometry? Our work will include but not limit to this case. We begin with a consequence of our main theorem which implies the converse Gluing Theorem.

A gluing is said by isometry if any two glued paths have the same length. In our assumptions, we allow the gluing of multiple but finitely many spaces, and also allow one point to be identified with multiple points. In this paper, volume means the Hausdorff volume of the top dimension. Due to [LR1], our statements stay true if replace Hausdorff volume by rough volume.

Theorem A (Characterization of volume preserving gluing). *If a gluing of finitely many n -dimensional Alexandrov spaces $\{Z_\alpha\}$ produces an Alexandrov space X without losing n -dimensional Hausdorff volume, then*

- (A.1) *the gluing is along the boundaries of $\{Z_\alpha\}$;*
- (A.2) *the gluing is by isometry;*
- (A.3) *each point is glued with finitely many points;*
- (A.4) *in any neighborhood of a gluing point in Z_α , the set of points which is glued with only one point has the same dimension as the boundary;*
- (A.5) *the set of points which is glued with more than one point has at least 1 lower dimension than the boundary.*

In Theorem A, (A.3)–(A.5) are the consequence of (A.1) and (A.2). The gluing which does not satisfy any one of (A.1)–(A.5) can not produce an Alexandrov space (Example 1.9). By (A.4) and (A.5), the gluing is completely determined by the part of one-to-one gluing. The gluing along non-extremal subset is allowed (Example 1.12).

Theorem A implies the converse gluing theorem we mentioned earlier. Together with Theorem 0.2, we have

Corollary 0.3. *Assume that n -dimensional Alexandrov spaces X and Y are glued via an identification $x \sim \phi(x)$, where $\phi : \partial X \rightarrow \partial Y$ is a one-to-one map with $\phi(\partial X) = \partial Y$. Then the glued space is an Alexandrov space if and only if ϕ is an isometry.*

In general, conditions (A.1)–(A.5) are not sufficient to guarantee the glued space being an Alexandrov space (Example 1.12). As a generalization of Theorem 0.2 and Corollary 0.3, we conjecture that

Conjecture. *A volume preserving gluing of n -dimensional Alexandrov spaces produces an Alexandrov space if and only if the gluing is by isometry and the induced gluing of spaces of directions produces Alexandrov spaces with curvature bounded from below by 1.*

We now state our main theorem and then discuss its relation with Theorem B. We call a map shrinking if it is surjective and 1-Lipschitz (distance non-increasing).

Theorem B (Shrinking rigidity Theorem). *Let X be an n -dimensional Alexandrov space and $Z = \coprod Z_\alpha$ be the disjoint union of finitely many n -dimensional Alexandrov spaces. A shrinking map $f : Z \rightarrow X$ preserves Hausdorff volume in the same dimension if and only if X is isometric to a space glued from $\{Z_\alpha\}$ which satisfies (A.1)–(A.5), and f is the projection map which preserves the length of path.*

In Theorem A, by assuming that X is glued from $\{Z_\alpha\}$, one can view X as a quotient space equipped with the induced length metric. Then the projection map $f : Z \rightarrow X$ is naturally shrinking and “gluing by isometry” is equivalent to that f preserves the length of path. Thus Theorem B is a generalization of Theorem A. We would like to point out that, even with the volume condition, “shrinking” implying “gluing” still requires a lot of hard work and the Alexandrov structures are crucially involved.

Theorem B can be viewed as a rigidity theorem for relatively maximum volume. Let

$$\mathfrak{A}\left(\{Z_\alpha\}_{\alpha=1}^{N_0}\right) = \{X \in \text{Alex}^n(\kappa) : \text{there is a 1-Lipschitz onto map } f : \coprod_{\alpha=1}^{N_0} Z_\alpha \rightarrow X\}.$$

Then $\text{vol}(X) \leq \sum_{\alpha=1}^{N_0} \text{vol}(Z_\alpha)$. By Theorem B, for any $X \in \mathfrak{A}$ with $\text{vol}(X) = \sum_{\alpha=1}^{N_0} \text{vol}(Z_\alpha)$, X is isometric to $\coprod_{\alpha=1}^{N_0} Z_\alpha$ up to a gluing which follows (A.1)–(A.5). In this sense, Theorem B verifies and generalizes Conjecture 0.1 in [LR2]. X may not be unique up to its topological type even when assuming that X can not be glued any more (Example 1.11). Theorem A and B are both applied to the self-gluing, i.e., Z has only one component (Example 1.11 and 1.13). From Theorem B and its proof, one can get the following easy corollaries, which may be useful elsewhere.

Corollary 0.4. *Let Z, X be n -dimensional Alexandrov spaces and $\Omega \subset Z$ be an open subset with $\Omega \cap \partial Z = \emptyset$. Assume $f : \Omega \rightarrow X$ is a volume preserving 1-Lipschitz map, then f is a local isometry.*

Corollary 0.5. *Under the assumption in Theorem B, if any of the following is satisfied, then Z has only one component and f is an isometry.*

- (i) f is injective.
- (ii) $\partial Z_\alpha = \emptyset$ for some α .
- (iii) $f(\partial Z) \subseteq \partial X$.

Theorem A and B are not true without assuming the Alexandrov structure. For example, X can be the space by shrinking (without identification) the metric over any zero measure subset of an Alexandrov space Z (see Example 1.5), and the shrinking map $f : Z \rightarrow X$ is still volume preserving. In particular, f is one-to-one but the metric on X is not induced by any gluing of Z . This is also an example for Corollary 0.5 (i), which is equivalent to that if one shrinks the metric over a zero measure subset in an Alexandrov space without identification, then the new space is no longer an Alexandrov space.

The starting point of our proof is a volume formula for “ ϵ -ball tubes” (Lemma 2.7). We first show that $f : Z \rightarrow X$ is bi-Lipschitz and almost preserves length of path when restricted to the set of (n, δ) -strained points (the points whose small neighborhood is almost isometric to a region of \mathbb{R}^n), where n is the dimension of Z_α . The main difficulty is that, the almost length preserving does not naturally approach to a length preserving by taking limit, because curve convergence does not imply length convergence in general. One would never succeed to do so without using the lower curvature bound (Example 1.5). A basic tool to overcome this is the Dimension comparison Lemma (Lemma 3.7) which essentially relies on the triangle comparison for lower curvature bound.

We divide the paper into five sections. In *Section 1*, we reformulate our main results in a technique way (Theorem 1.1), and provide examples for various gluing. We will also give an outline for the proof.

In *Section 2*, we recall properties for singular points in Alexandrov spaces from [BGP] and [OS]. A volume formula for “ ϵ -ball tubes” (Lemma 2.7) and the Almost Maximum Volume Theorem (Theorem 2.8) will be established for the later use.

Section 3 is the first attack to the Main Theorem. Using the tools from Section 2, we mainly show that the volume preserving 1-Lipschitz onto map is an isometry when restricted to the interior and subject to the intrinsic metric (Lemma 1.6).

In *Section 4*, we complete the proof of the Theorem 1.1. The main effort is to show that f preserves the length of path (Lemma 1.8).

In *Section 5*, we will give some applications, regarding the shrinking of space of directions (Theorem 5.3) and the relatively maximum/almost maximum volume (Theorem 5.4 and 5.5) in Alexandrov geometry. The second topic was discussed in [LR2]. Some theorems were under extra conditions. Here we give simple proofs using Theorem B without extra assumption. This work is a natural extension of Grove and Petersen's work [GP] in Riemannian geometry.

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1. SHRINKING RIGIDITY THEOREM AND EXAMPLES

Conventions and notations

- $\text{vol}(A)$ — the n -dimensional Hausdorff volume of A , where n is the Hausdorff dimension.
- $d_A(x, y) = |xy|_A$ — the distance between two points x and y subject to the length metric of A .
- ∂A — the boundary of A .
- $A^\circ = A - \partial A$ — the interior of A .
- $\dim_H(A)$ — the Hausdorff dimension of A .
- $\dim_T(A)$ — the topological dimension of A .
- $\dim(A)$ — both Hausdorff dimension and topological dimension of A .
- $B_r(p)$ — the metric ball $\{x \in X : |px| < r\}$.
- $[pq]_X$ — a minimal geodesic joint points p and q in X .
- $\tau(\delta)$ — a function of δ with $\lim_{\delta \rightarrow 0} \tau(\delta) = 0$. Without declaration, $\tau(\delta)$ is independent on the selection of points.
- $X_n \xrightarrow{d_{GH}} X$ — the sequence X_n Gromov-Hausdorff converges to X .
- Let $A \subset X$ and $f : X \rightarrow Y$. We call the restricted map $f|_A$ an isometry if $|ab|_A = |f(a)f(b)|_{f(A)}$ for any $a, b \in A$.

By $\text{Alex}^n(\kappa)$ we will denote the class of n -dimensional Alexandrov spaces with curvature $\geq \kappa$. For $X \in \text{Alex}^n(\kappa)$, we will use the following notations (c.f. [BGP]).

- $\Sigma_x(X)$ or sometimes Σ_x — the space of directions for a point $x \in X$.
- X^δ — the (n, δ) -strained points in X .
- $\text{sn}_\kappa(t) = \begin{cases} \frac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa} t), & \text{if } \kappa > 0; \\ t, & \text{if } \kappa = 0; \\ \frac{1}{\sqrt{-\kappa}} \sinh(\sqrt{-\kappa} t), & \text{if } \kappa < 0. \end{cases}$

We begin with an exact definition of the gluing of length metric spaces ([BBI] §3). Let $\{(Z_\alpha, d_\alpha)\}$ be a collection of compact *length metric spaces*. The distance function d on the disjoint union $Z = \coprod Z_\alpha$ is defined by $d(x, y) = d_\alpha(x, y)$ if $x, y \in Z_\alpha$ for some α , otherwise $d(x, y) = \infty$. Let R be an equivalence relation (denoted as $x \stackrel{R}{\sim} y$) over Z . The quotient

semi-metric d_R on Z is defined as

$$d_R(x, y) = \inf \left\{ \sum_{i=1}^N d(p_i, q_i) : p_1 = x, q_N = y, q_i \stackrel{R}{\sim} p_{i+1}, N \in \mathbb{N} \right\}.$$

Then the quotient space $(Z/d_R, \overline{d_R})$ is a length metric space, where $\overline{d_R}$ is the induced metric. We call $(Z/d_R, \overline{d_R})$ glued from Z (or $\{(Z_\alpha, d_\alpha)\}$) along the equivalence relation R . The induced projection map $f : Z \rightarrow X$ is a 1-Lipschitz onto.

We now formulate our Theorem 1.1. Let $Z = \coprod_{\alpha=1}^{N_0} Z_\alpha$ be the disjoint union of compact length metric spaces $\{Z_\alpha\}$. Let $f : Z \rightarrow X$ be a shrinking (1-Lipschitz onto) map which preserves volume ($\text{vol}(X) = \text{vol}(Z) = \sum_{\alpha=1}^{N_0} \text{vol}(Z_\alpha)$). We would like to study the characterization of f and the metric on X under the assumption that Z_α and X are all Alexandrov spaces. Note that if assume a distance non-decreasing map $g : X \rightarrow Z$ with $\text{vol}(X) = \text{vol}(Z)$, one can extend g^{-1} to a shrinking map $f : Z \rightarrow X$ using the compactness of X and Z_α . Thus this is equivalent to the above setup.

Let $\partial Z = \coprod_{\alpha=1}^{N_0} \partial Z_\alpha$ denote the disjoint union of boundaries and $Z^\circ = Z - \partial Z$ denote the interior points of Z . For $x \in X$, $f^{-1}(x)$ may not be unique. We will show that X is isometric to a space glued from Z and f is the induced projection map. Then in the sense of gluing, $f^{-1}(x)$ will be identified as one point. We let $G_X = \{x \in X : f^{-1}(x) \text{ is not unique}\}$ and $G_Z = f^{-1}(G_X) \subset Z$. We give a stratification of these points. Let

$$G_X^m = \{x \in X, f^{-1}(x) \text{ contains exactly } m \text{ points}\},$$

and $G_Z^m = f^{-1}(G_X^m) \subseteq Z$. We call $m_0 = \max\{m : G_Z^m \neq \emptyset\}$ the maximum gluing number. In general, m_0 is independent of N_0 . We will show that $m_0 \leq C(n, \kappa, \text{diam}(Z_\alpha), \text{vol}(Z_\alpha)) < \infty$. Clearly, $G_X = \bigcup_{m=2}^{m_0} G_X^m$ and $G_Z = \bigcup_{m=2}^{m_0} G_Z^m$.

Theorem 1.1 (Shrinking rigidity). *Let $X \in \text{Alex}^n(\kappa)$ and $Z = \coprod_{\alpha=1}^{N_0} Z_\alpha$, where $Z_\alpha \in \text{Alex}^n(\kappa)$, $\alpha = 1, \dots, N_0$. If $\text{vol}(X) = \text{vol}(Z)$ and there exists a shrinking map $f : Z \rightarrow X$, then X is isometric to a space glued from $\{Z_\alpha\}_{\alpha=1}^M$ and f is the projection map induced by the gluing. Moreover,*

- (i) if $G_Z \neq \emptyset$ then $G_Z \subseteq \partial Z$;
- (ii) f preserves the length of path. Consequently, the gluing is by isometry and $f|_{Z_\alpha^\circ}$ is an isometric embedding;
- (iii) $m_0 \leq \frac{\text{vol}(B_{d_0}(S_\kappa^n))}{v_0}$, where $d_0 = \max\{\text{diam}(Z_\alpha)\}$ and $v_0 = \min_{\alpha} \{\text{vol}(Z_\alpha)\}$;
- (iv) if $G_Z \neq \emptyset$, then for any $\hat{p} \in G_Z$, $p = f(\hat{p}) \in G_X$ and $r > 0$,

$$\dim(B_r(\hat{p}) \cap G_Z^2) = \dim(B_r(p) \cap G_X^2) = n - 1$$

and

$$\dim_H \left(\bigcup_{m=3}^{m_0} G_Z^m \right) = \dim_H \left(\bigcup_{m=3}^{m_0} G_X^m \right) \leq n - 2.$$

Remark 1.2.

(1.2.1) By evenly cutting $X = \mathbb{S}_1^n$ into m_0 petals (Z_α) with diameter 1, we see that the estimate in Theorem 1.1(iii) is sharp for the gluing of multiple spaces. We also have $\dim(G_Z^2) = \dim(G_X^2) = n - 1$, $G_Z^m = \emptyset$ for $3 \leq m \leq m_0 - 1$ and $\dim(G_Z^{m_0}) = \dim(G_X^{m_0}) = 1$, where $G_X^{m_0}$ is the common diameter glued with m_0 petals.

The above example has a specialty that $m_0 = N_0$. For $N_0 = 1$ (self-gluing), an example (Example 1.14) for $m_0 = \frac{1}{v_0} \cdot \text{vol}(B_{d_0}(S_\kappa^n)) = 2$ is constructed. However, the author did not succeed to find an example of self-gluing with $m_0 = \frac{1}{v_0} \cdot \text{vol}(B_{d_0}(S_\kappa^n)) \geq 3$.

(1.2.2) When Z has only one component ($N_0 = 1$), the Shrinking rigidity Theorem is applied for a self-gluing (see Example 1.11, 1.13 for Theorem 1.1(iii) and (iv)). A special case in this kind can be further classified (Theorem 5.4 and Theorem 5.5).

In some special cases, we can determine that f is in fact an isometry.

Corollary 1.3. *Under the assumptions as in Theorem 1.1, if any of the following is satisfied, then $N_0 = 1$ and f is an isometry.*

- (i) $\partial Z_\alpha = \emptyset$ for some $1 \leq \alpha \leq N_0$.
- (ii) $G_Z = \emptyset$.
- (iii) $G_X \subseteq \partial X$.
- (iv) $f(\partial Z) \subseteq \partial X$.
- (v) $f^{-1}(X^\delta) \cap G_Z = \emptyset$ for $\delta > 0$ small.
- (vi) $f^{-1}(X^\delta) \subseteq Z^\circ$ for $\delta > 0$ small.

Proof. (i) and (ii) are trivial by Theorem 1.1(ii). (iv) follows by (iii) and (vi) follows by (v) with the fact $G_Z \subseteq \partial Z$. We first prove (iii). By the assumption, we have $f^{-1}(X^\circ) \subseteq Z - G_Z$. By Theorem 1.1(ii), $f|_{Z-G_Z}$ is an isometry. Consequently, f is an isometry since X° is totally geodesic in X .

To prove (v) by (iii), it's sufficient to show that $G_X \subseteq \partial X$. For any $x \in G_X$, by Theorem 1.1(iv), we have

$$\dim_H(B_r(x) \cap G_X) = n - 1.$$

By the assumption, we see that $G_X \cap X^\delta = \emptyset$. Then $B_r(x) \cap G_X \subseteq B_r(x) \setminus X^\delta$ and thus

$$\dim_H(B_r(x) \setminus X^\delta) \geq \dim_H(B_r(x) \cap G_X) = n - 1.$$

If $x \notin \partial X$, then there is $r > 0$ so that $B_r(x) \subset X^\circ$. Thus by [BGP] 10.6.1,

$$\dim_H(B_r(x) \setminus X^\delta) \leq \dim_H(X^\circ \setminus X^\delta) \leq n - 2,$$

a contradiction. □

If X is a space glued from $\{Z_\alpha\}$, we expect that the space of directions $\Sigma_x(X) \in \text{Alex}^{n-1}(1)$ is also glued from $\Sigma_{\hat{z}_\beta}(Z)$, where $\{\hat{z}_\beta\} = f^{-1}(x) \subset Z$ are the pre-images. Let $\Sigma_{f^{-1}(x)} = \coprod_{\beta} \Sigma_{\hat{z}_\beta}(Z_\beta)$ denote the disjoint union of spaces of directions $\Sigma_{\hat{z}_\beta} \in \text{Alex}^{n-1}(1)$.

Theorem 1.4 (Gluing of spaces of directions). *Under the assumption as in Theorem 1.1, for any $x \in X$, Σ_x is a space glued from $\Sigma_{f^{-1}(x)}$ without losing volume.*

Our proof of Theorem 1.1 starts from a volume formula (Lemma 2.7) of “ ϵ -ball tubes”, which connects the volume of a tubular-like neighborhood of path γ with its length $L(\gamma)$. Due to the singularity of Alexandrov spaces, this formula can be established only for (n, δ) -strained points, and thus the usage of this formula is also limited. Let $\tau(\delta)$ denote a function of δ with $\lim_{\delta \rightarrow 0} \tau(\delta) = 0$. Together with the properties $f(Z^\delta) \subseteq X^{\tau(\delta)}$ (Lemma 3.1), $Z^\delta \cap G_Z = \emptyset$ (Lemma 3.2) and that $f|_{Z^\delta}$ is bi-Lipschitz (Lemma 3.3), we are able to show that f almost preserves the length of path that only contains points in Z^δ (Lemma 3.4).

The main difficulty is to extend it to an exact length preserving for any path in Z . In general, given a path $\gamma \in Z$, using the structure of Z^δ and $X^{\tau(\delta)}$, one can find $\gamma_\delta \subset Z^\delta$ such that $\gamma_\delta \rightarrow \gamma$, $f(\gamma_\delta) \rightarrow f(\gamma)$ with $L(\gamma_\delta) \rightarrow L(\gamma)$ and $L(\gamma_\delta) = L(f(\gamma_\delta)) + \tau(\delta)$. However, these are not sufficient to imply $L(f(\gamma_\delta)) \rightarrow L(f(\gamma))$, as $\delta \rightarrow 0$. Without this, one can not conclude that $L(\gamma) = L(f(\gamma))$. See the following example.

Example 1.5 (The shrinking cube). Let Z be a unit n -dimensional cube ($n \geq 3$). Let X be the same cube in which the length of one edge $[AB]$ is redefined to be $\frac{1}{2}$. The new length metric (on X) is “smaller” than the Euclidean metric. Let $f : Z \rightarrow X$ be the identity map. Note that $f([AB])$ is the shrunk edge in X . Then $L([AB]) = 1$ and $L(f([AB])) = \frac{1}{2}$. For any path $\gamma_i \rightarrow [AB]$ with $\gamma_i \cap [AB] = \emptyset$, $L(f(\gamma_i)) = L(\gamma_i)$ and

$$\liminf_{i \rightarrow \infty} L(f(\gamma_i)) = \liminf_{i \rightarrow \infty} L(\gamma_i) \geq L([AB]) = 1 > \frac{1}{2} = L(f([AB])).$$

Our approach is to first show that the gluing only occurs within the boundary (Lemma 3.8 (i)) using the almost length preserving. This implies that $f(Z^\circ)$ is open in X . Together with the Dimension comparison Lemma (technique Lemma 3.7) for Alexandrov spaces, we find an approximation $\gamma_\delta \subset Z^\delta$ such that $\gamma_\delta \rightarrow \gamma$, $L(\gamma_\delta) \rightarrow L(\gamma)$ and $f(\gamma_\delta) \rightarrow f(\gamma)$, $L(f(\gamma_\delta)) \rightarrow L(f(\gamma))$ as $\delta \rightarrow 0$ for any given path $\gamma \subset Z^\circ$ (Lemma 3.9). This allows us to prove that f is an isometric embedding when restricted to the interior of Z (Lemma 1.6). Note that this idea would not work for $\gamma \subset \partial Z$, because Lemma 3.9 is not true in this case (Example 3.6).

Lemma 1.6 (Internal isometry). *Let the assumption be as in Theorem 1.1. Then $G_Z \subseteq \partial Z$ and $f|_{Z^\circ}$ is an isometry.*

Using the above lemma, we are able to establish the gluing structures of the spaces of directions (Theorem 1.4), which will imply the following Gluing dimension Lemma by Lemma 3.8 (2).

Lemma 1.7 (Gluing dimension). *Let the assumption be as in Theorem 1.1. Then for any $\hat{p} \in G_Z$, $p = f(\hat{p}) \in G_X$ and $r > 0$,*

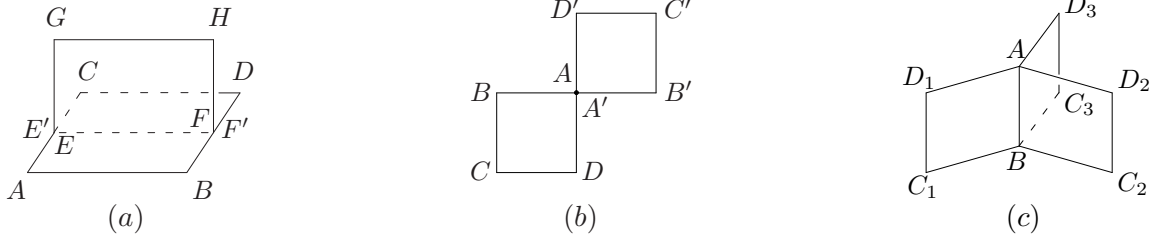
$$\dim(B_r(\hat{p}) \cap G_Z) = \dim(B_r(p) \cap G_X) = n - 1.$$

In the last step, we prove Theorem 1.1(ii). The proof relies on the local structure of almost conic gluing (Lemma 4.1) and the first variation formula.

Lemma 1.8 (Length preserving). *Let the assumption be as in Theorem 1.1. Then f preserves the length of path.*

We complete this section by giving some examples for various kinds of gluing.

Example 1.9 (Non-Alexandrov gluing). The following examples are not Alexandrov spaces, since one can find bifurcated geodesics near the glued points. In (a), rectangle $ABCD$ is glued with rectangle $EFGH$ along the interior segment $[E'F']$ and the edge $[EF]$. This gluing does not satisfy (A.1). In (b), square $ABCD$ is glued with square $A'B'C'D'$ at the point $A \sim A'$. This gluing does not satisfy (A.5). In (c), three rectangles are glued along an edge $[AB]$ with equal length. This gluing does not satisfy (A.4).



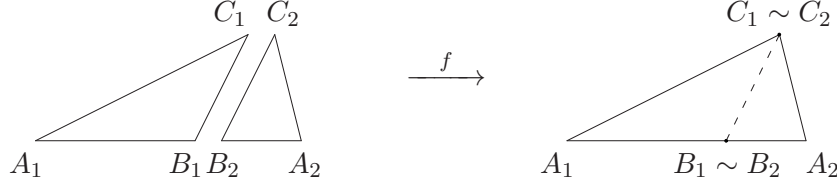
Example 1.10 (Non-isometric gluing). Let $A(r)$ denote the 2-dimensional Euclidean square with side length r . Consider the boundary gluing of $A(r)$ and $A(R)$. Let $\phi : \partial A(r) \rightarrow \partial A(R)$ be a map which preserves the central angle. Then $X = A(r) \amalg A(R)/x \sim \phi(x)$ is the glued space via the identification $x \sim \phi(x)$. If $r = R$, then ϕ is an isometry, and thus X is an Alexandrov space as a doubled square. On the other hand, Corollary 0.2 concludes that if $X \in \text{Alex}^2(\kappa)$, then ϕ has to be an isometry, i.e., $r = R$. In fact, if $r < R$, let $a, b \in f(\partial A(R))$ and $c \in f(A(R)^\circ)$ near b , where $f : A(r) \amalg A(R) \rightarrow X$ is the projection map. Then geodesics $[ab]_X$ and $[ac]_X$ have overlaps, which yields a geodesic bifurcation.

One can also construct a similar example for the boundary gluing of two disks with radius r and R . By Corollary 0.2, such gluing produces an Alexandrov space if and only if $r = R$. Note that there is no geodesic bifurcation in the case $r \neq R$. These are also examples for (A.2).

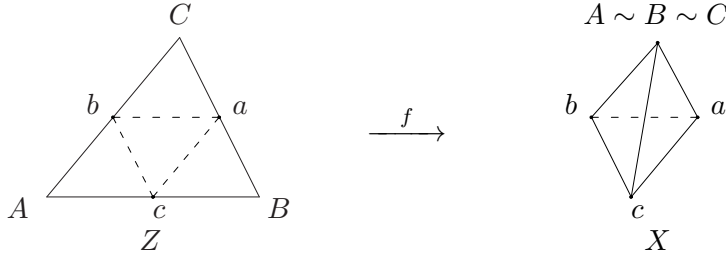
Example 1.11 (Involutional self-gluing). This is an example for self-gluing (c.f. [GP]). Let $Z = \mathbb{D}^2$ be a 2-dimensional flat unit disk. Then $\partial Z = \mathbb{S}^1(1)$ is a unit circle. Let $\phi : \partial Z \rightarrow \partial Z$ be a map and $X = \mathbb{D}^2/x \sim \phi(x)$ be the glued space. By Theorem 5.4, X is an Alexandrov space if and only if ϕ is a reflection, antipodal map or identity, where X is homeomorphic to \mathbb{S}^2 , $\mathbb{R}P^2$ and \mathbb{D}^2 respectively. From the construction, we see that the maximum gluing number $m_0 \leq 2$. However, if we estimate using Theorem 1.1(iii), we get

$$m_0 \leq \frac{\pi \cdot 2^2}{\pi \cdot 1^2} = 4.$$

Example 1.12 (Gluing along non-extremal subset). When glue Alexandrov spaces along non-extremal subsets, it may still produce an Alexandrov space. In the following gluing of two flat triangle planes, where $\angle A_1B_1C_1 + \angle A_2B_2C_2 = \pi$ and edge $[B_1C_1]$ is glued with edge $[B_2C_2]$. The glued space is also a triangle. When $\angle A_1B_1C_1 > \frac{\pi}{2}$, edge $[B_1C_1]$ is not an extremal subset in the triangle plane $\triangle A_1B_1C_1$. If $\angle A_2B_2C_2 + \angle A_1B_1C_1 > \pi$, then the glued space is not convex, thus it is not an Alexandrov space.



Example 1.13 (Three points gluing in a self-gluing). This is an example for self-gluing with $m_0 \geq 3$. Let Z be a triangle. We identify points on each side via a reflection about the mid point, i.e., $[Ab] \sim [Cb]$, $[Ac] \sim [Bc]$, $[Ba] \sim [Ca]$ and thus $A \sim B \sim C$ are glued to one point. The glued space X is a tetrahedron, which belongs to $\text{Alex}^2(0)$. We see that $G_Z^2 = [AB] \cup [BC] \cup [AC] \setminus \{A, B, C, a, b, c\}$ is dense in ∂Z , $\dim(G_Z^2) = 1$ and $G_Z^3 = \{A, B, C\}$ is isolated.



Example 1.14. (Maximum gluing number) Given $\kappa > 0$, let $X = B_{\frac{\pi}{2\sqrt{\kappa}}}(\mathbb{S}_1^n)$ be the semi n -sphere. By Theorem 1.1(iii), the maximum gluing number $m_0 \leq \frac{\text{vol}(B_{\pi/\sqrt{\kappa}}(\mathbb{S}_1^n))}{\text{vol}(X)} = 2$, which states that any 3 points gluing will not result in an Alexandrov space. This is also verified by Theorem 5.4, in the case $\Sigma_p = \mathbb{S}_1^{n-1}$ and $R = \frac{\pi}{2\sqrt{\kappa}}$.

2. PRELIMINARIES

We first recall properties for local structures in Alexandrov spaces. Let $X \in \text{Alex}^n(\kappa)$. For any (n, δ) -strained point $p \in X$, the metric near p is almost the same as the Euclidean metric. By $X^\delta(\rho)$ we denote the collection of points with (n, δ) -strainers $\{(a_i, b_i)\}_{i=1}^n$ of size $\rho > 0$, where $\rho = \min_{1 \leq i \leq n} \{|pa_i|, |pb_i|\} > 0$.

Theorem 2.1 ([BGP] Theorem 9.4). *Let $X \in \text{Alex}^n(\kappa)$. If $p \in X^\delta(\rho)$, then the map $\psi : X \rightarrow \mathbb{R}^n$, $x \mapsto (|a_1x|, \dots, |a_nx|)$ maps a small neighborhood U of p $\tau(\delta, \delta_1)$ -almost isometrically onto a domain in \mathbb{R}^n , i.e., $||\psi(x)\psi(y)| - |xy|| < \tau(\delta, \delta_1)|xy|$ for any $x, y \in U$, where $\delta_1 = \rho^{-1} \cdot \text{diam}(U)$. In particular, for $\epsilon \ll \delta\rho$ small, ψ is an $\tau(\delta)$ -almost isometry when restricting to $B_\epsilon(p)$.*

Let $X^{(m, \delta)}$ denote the collection of (m, δ) -strained points, $m = 1, 2, \dots, n-1$. The following two lemmas give a description of the local structure near the points in $X^{(n-1, \delta)}$.

Theorem 2.2 ([BGP] 12.8). *Let $X \in \text{Alex}^n(\kappa)$. For any $p \in X^{(n-1, \delta)}$, if $p \in X^\circ$, then $p \in X^{\tau(\delta)}$.*

Theorem 2.3 ([BGP] 12.9.1). *Let $X \in \text{Alex}^n(\kappa)$ and $p \in X^{(n-1, \delta)}$ with the strainer size ρ . If $p \in \partial X$, then a neighborhood U of p is $\tau(\delta, \delta_1)$ -isometric mapped onto a cube in \mathbb{R}^n , where $\delta_1 = \rho^{-1} \cdot \text{diam}(U)$. Moreover, $U \cap \partial X$ maps onto one of the hyperfaces of this cube.*

The points which do not admit $(n-1, \delta)$ -strainer have dimension $\leq n-2$. Moreover, we have

Theorem 2.4 ([BGP] 10.6). *Let $X \in \text{Alex}^n(\kappa)$. For $1 \leq m \leq n$ and sufficiently small $\delta > 0$, $\dim_H(X \setminus X^{(m, \delta)}) \leq m-1$.*

A consequence of Theorem 2.2 and 2.4 is that

Corollary 2.5 ([BGP] 10.6.1). *Let $X \in \text{Alex}^n(\kappa)$. For sufficiently small $\delta > 0$, $\dim_H(X^\circ \setminus X^\delta) \leq n-2$.*

Let $X^{\text{Reg}} = \bigcap_{\delta > 0} X^\delta$. It's not hard to see that for any $p \in X^{\text{Reg}}$, $\Sigma_p = \mathbb{S}_1^{n-1}$. Note that X^{Reg} is dense in X . Moreover,

Theorem 2.6 ([OS]). *Let $X \in \text{Alex}^n(\kappa)$. Then $\dim_H(X \setminus X^{\text{Reg}}) \leq n-1$.*

We now consider the volume of small balls in an Alexandrov space. A consequence of Theorem 2.1 is that for any $p \in X^\delta(\rho)$ and $\epsilon \ll \delta\rho$,

$$\text{vol}(B_\epsilon(p)) = (1 + \tau(\delta)) \cdot \text{vol}(B_\epsilon(\mathbb{R}^n)) = (1 + \tau(\delta)) \cdot \text{vol}(\mathbb{S}_1^{n-1}) \int_0^\epsilon t^{n-1} dt.$$

Furthermore, let x_1, x_2, \dots, x_{N+1} be $N+1$ points in $X^\delta(\rho)$. There is an estimate of the volume of the “ ϵ -ball tube” $\bigcup_{i=1}^{N+1} B_\epsilon(x_i)$, in terms of ϵ and $\sum_{i=1}^N |x_i x_{i+1}|$ with a higher order error.

Lemma 2.7 (Volume of an ϵ -ball tube¹, [LR2] Lemma 1.4). *Let $X \in \text{Alex}^n(\kappa)$ and $x_i \in X^\delta(\rho)$, $i = 1, 2, \dots, N+1$ satisfy that $0 < |x_i x_{i+1}| < 2\epsilon \ll \delta\rho$ and $B_\epsilon(x_i) \cap B_\epsilon(x_j) \cap B_\epsilon(x_k) = \emptyset$ for $i \neq j \neq k \neq i$. Then the volume of the ϵ -ball tube $\bigcup_{i=1}^{N+1} B_\epsilon(x_i)$ (see Figure 1) satisfies:*

$$\begin{aligned} (2.1) \quad & (1 + \tau(\delta)) \cdot \text{vol}\left(\bigcup_{i=1}^{N+1} B_\epsilon(x_i)\right) \\ &= \text{vol}(B_\epsilon(\mathbb{R}^n)) + 2\epsilon \cdot \text{vol}(B_\epsilon(\mathbb{R}^{n-1})) \sum_{i=1}^N \int_{\theta_i}^{\frac{\pi}{2}} \sin^n(t) dt, \end{aligned}$$

where $\theta_i \in [0, \frac{\pi}{2}]$ such that $\cos \theta_i = \frac{|x_i x_{i+1}|}{2\epsilon}$. If in addition, $|x_i x_{i+1}| \leq 2\epsilon^2$ for all $1 \leq i \leq N$, then

$$\begin{aligned} (2.2) \quad & (1 + \tau(\delta)) \cdot \text{vol}\left(\bigcup_{i=1}^{N+1} B_\epsilon(x_i)\right) \\ &= \text{vol}(B_\epsilon(\mathbb{R}^n)) + \text{vol}(B_\epsilon(\mathbb{R}^{n-1})) \sum_{i=1}^N |x_i x_{i+1}| + O(\epsilon^{n+1}) \sum_{i=1}^N |x_i x_{i+1}|, \end{aligned}$$

¹This may be viewed as a special case of the co-area formula for Alexandrov spaces.

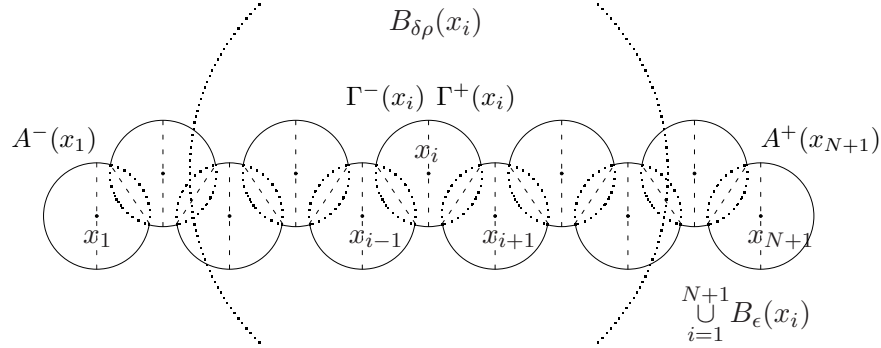


Figure 1

We complete this section by establishing a theorem about the almost (absolute) maximum volume.

Theorem 2.8 (Almost Maximum Volume). *Let $X \in \text{Alex}^n(1)$. If $\text{vol}(X) \geq \text{vol}(\mathbb{S}_1^n) - \epsilon$, then there is a $\tau(\epsilon)$ -onto $h : X \rightarrow \mathbb{S}_1^n$, which is $\tau(\epsilon)$ -almost isometry as well. In particular, if $p \in A \in \text{Alex}^n(\kappa)$ and $\text{vol}(\Sigma_p) \geq \text{vol}(\mathbb{S}_1^{n-1}) - \delta$, then $p \in A^{\tau(\delta)}$.*

Proof. We first inductively define a distance non-decreasing map $h_n : X \rightarrow \mathbb{S}_1^n$. The case for $n = 1$ is trivial. Let $p \in X$, then $\Sigma_p \in \text{Alex}^{n-1}(1)$. Assume $h_{n-1} : \Sigma_p \rightarrow \mathbb{S}_1^{n-1}$ is defined and is distance non-increasing, then $h_n = (h_{n-1}, \text{id}) \circ \exp_p^{-1}$ is defined via the composition (c.f. [BGP] 10.2):

$$X \xrightarrow{\exp_p^{-1}} C_1^\pi(X_p) \xrightarrow{(h_{n-1}, \text{id})} C_1^\pi(\mathbb{S}_1^{n-1}) = \mathbb{S}_1^n,$$

where C_1^π is the spherical suspension. Clearly h_n is also distance non-increasing. Let $h = h_n$ and $\Omega = \mathbb{S}_1^n - h(X)$. We have

$$\text{vol}(\Omega) = \text{vol}(\mathbb{S}_1^n) - \text{vol}(h(X)) \leq \text{vol}(\mathbb{S}_1^n) - \text{vol}(X) < \epsilon.$$

Let $B_r \subset \mathbb{S}_1^n$ be the metric ball which is not contained in $h(X)$, i.e., $B_r \subseteq \Omega$. Then

$$\epsilon > \text{vol}(\Omega) \geq \text{vol}(B_r) = \text{vol}(\mathbb{S}_1^{n-2}) \cdot \int_0^r \sin^{n-2}(t) dt.$$

Thus $r < \tau(\epsilon)$ and h is a $\tau(\epsilon)$ -onto.

We now show that h is a $\tau(\epsilon)$ -isometry. Let $p, x \in X$ and $\tilde{p} = h(p)$, and $\tilde{x} = h(x) \in \mathbb{S}_1^n$. It's clear that $|\tilde{p}\tilde{x}|_{\mathbb{S}_1^n} \geq |px|_X$. Let q be a point in X such that $|pq|_X = \sup_{t \in X} \{|pt|_X\} = L$ and $\tilde{q} = h(q) \in \mathbb{S}_1^n$. Because

$$\text{vol}(\mathbb{S}_1^n) - \epsilon \leq \text{vol}(X) \leq \text{vol}(B_L(\mathbb{S}_1^n)),$$

we have $L \geq \pi - \tau(\epsilon)$. On the other hand,

$$\begin{aligned} 2\pi &\geq |\tilde{p}\tilde{x}|_{\mathbb{S}_1^n} + |\tilde{p}\tilde{q}|_{\mathbb{S}_1^n} + |\tilde{x}\tilde{q}|_{\mathbb{S}_1^n} \geq |\tilde{p}\tilde{x}|_{\mathbb{S}_1^n} + |pq|_X + |xq|_X \\ &\geq |\tilde{p}\tilde{x}|_{\mathbb{S}_1^n} + |pq|_X + (|pq|_X - |px|_X) = |\tilde{p}\tilde{x}|_{\mathbb{S}_1^n} + 2L - |px|_X. \end{aligned}$$

Thus $|\tilde{p}\tilde{x}|_{\mathbb{S}_1^n} - |px|_X \leq 2\pi - 2L < \tau(\epsilon)$. □

3. INTERNAL ISOMETRY AND BOUNDARY GLUING

In this section, we will prove Lemma 1.6, 1.7 and Theorem 1.4. Except the Dimension comparison Lemma (Lemma 3.7), all lemmas and corollaries are under the assumptions as in Theorem 1.1. For a minimal geodesic $[pq]_X$ in X , we let $]pq[_X = [pq]_X \setminus \{p\}$, $]pq[_X = [pq]_X \setminus \{q\}$, $]pq[_X = [pq]_X \setminus \{p, q\}$. The following basic properties (Lemma 3.1 – 3.3) will be needed in the later proofs.

Lemma 3.1. $f(Z^\delta) \subseteq X^{\tau(\delta)}$. In particular, $f(Z^{\text{Reg}}) \subseteq X^{\text{Reg}}$.

Proof. Let $\hat{x} \in Z^\delta(\rho)$ and $x = f(\hat{x})$. For $\epsilon \ll \delta\rho$, because f is volume preserving and $f^{-1}(B_\epsilon(x)) \supseteq B_\epsilon(\hat{x})$, we have the following volume comparison:

$$\begin{aligned} \text{vol}(\Sigma_x) \cdot \int_0^\epsilon \text{sn}_\kappa^{n-1}(t) dt &\geq \text{vol}(B_\epsilon(x)) \\ &= \text{vol}(f^{-1}(B_\epsilon(x))) \geq \text{vol}(B_\epsilon(\hat{x})) \\ &= (1 - \tau(\delta)) \cdot \text{vol}(\mathbb{S}_1^{n-1}) \cdot \int_0^\epsilon t^{n-1} dt. \end{aligned}$$

By the almost maximum volume (Theorem 2.8), $x \in X^{\tau(\delta)}$. □

Recall that $G_X = \{x \in X : f^{-1}(x) \text{ is not unique.}\}$ and $G_Z = f^{-1}(G_X)$. To show Lemma 1.7, we need the property $G_Z \subseteq \partial Z$, i.e., the gluing occurs only along the boundaries. We first show that for $\delta > 0$ small, points in Z^δ are not glued with any other point, i.e., $G_Z \subseteq Z \setminus Z^\delta$.

Lemma 3.2. Let $d_0 = \max\{\text{diam}(Z_\alpha)\}$, $v_0 = \min\{\text{vol}(Z_\alpha)\}$. Then there is a constant $c = c(n, \kappa, d_0, v_0)$ such that for any $0 < \delta < c$, $Z^\delta \cap G_Z = \emptyset$. Consequently, $f(Z^\delta) = X \setminus f(Z \setminus Z^\delta)$ is open in X and for any $A \subseteq Z$, $f(A \setminus Z^\delta) = f(A) \setminus f(Z^\delta)$.

Proof. Argue by contradiction. Assume $f(\hat{x}_1) = f(\hat{x}_2) = x$ and $\hat{x}_1 \in Z^\delta$, $\hat{x}_2 \in Z_\alpha$. Let $d_\alpha = \text{diam}(Z_\alpha)$. By Lemma 3.1, $x \in X^{\tau(\delta)}$. Let $\epsilon > 0$ small such that $B_\epsilon(\hat{x}_1) \cap B_\epsilon(\hat{x}_2) = \emptyset$. By Bishop-Gromov relative volume comparison for Alexandrov spaces ([BBI], [LR2]), we have

$$\begin{aligned} 1 &= \frac{\text{vol}(f^{-1}(B_\epsilon(x)))}{\text{vol}(B_\epsilon(x))} \geq \frac{\text{vol}(B_\epsilon(\hat{x}_1)) + \text{vol}(B_\epsilon(\hat{x}_2))}{\text{vol}(B_\epsilon(x))} \\ &\geq \frac{\text{vol}(B_\epsilon(\hat{x}_1)) + \text{vol}(Z_\alpha) \cdot \frac{\int_0^\epsilon \text{sn}_\kappa^{n-1}(t) dt}{\int_0^{d_\alpha} \text{sn}_\kappa^{n-1}(t) dt}}{\text{vol}(B_\epsilon(x))} \\ &\geq \frac{(1 - \tau(\delta)) \cdot \text{vol}(\mathbb{S}_1^{n-1}) \cdot \int_0^\epsilon t^{n-1} dt + v_0 \cdot \frac{\int_0^\epsilon \text{sn}_\kappa^{n-1}(t) dt}{\int_0^{d_0} \text{sn}_\kappa^{n-1}(t) dt}}{(1 + \tau(\delta)) \cdot \text{vol}(\mathbb{S}_1^{n-1}) \cdot \int_0^\epsilon t^{n-1} dt}. \end{aligned}$$

Let $\epsilon \rightarrow 0$, we get

$$1 \geq \frac{(1 - \tau(\delta)) \cdot \text{vol}(\mathbb{S}_1^{n-1}) + \frac{v_0}{\int_0^{d_0} \text{sn}_\kappa^{n-1}(t) dt}}{(1 + \tau(\delta)) \cdot \text{vol}(\mathbb{S}_1^{n-1})}.$$

This is a contradiction for δ sufficiently small. □

Roughly speaking, we would like to cut curves in partitions and apply the volume formula (Lemma 2.7) to show that $f|_{Z^\delta}$ almost preserves length of paths. Let $\gamma \subset Z^\delta$ be a geodesic. We start from the piece-wise geodesics approximation for $f(\gamma)$ and apply the volume formula to this and the corresponding piece-wise geodesics in Z^δ simultaneously. We shall check that the corresponding piece-wise geodesics in Z^δ converge to γ .

Lemma 3.3. *For $\delta, \rho > 0$ small, let $\hat{x}_i \in Z^\delta(\rho)$ and $x_i = f(\hat{x}_i)$, $i = 1, 2$. There exists a constant $\epsilon = \epsilon(\delta, \rho) > 0$ such that if $|x_1 x_2|_X \leq \epsilon$, then $|f^{-1}(x_1) f^{-1}(x_2)|_Z \leq 2|x_1 x_2|_X$. Consequently, $f|_{Z^\delta}$ is bi-Lipschitz, and thus $f(Z^\delta) \subseteq X^{\tau(\delta)}$ is open and dense in X .*

Proof. Assume that $|x_1 x_2| = \epsilon \ll \delta \rho$ and $|f^{-1}(x_1) f^{-1}(x_2)| > 2\epsilon$. Consider the metric balls $B_\epsilon(x_1)$ and $B_\epsilon(x_2)$. By the volume formula (2.1),

$$\begin{aligned} & (1 + \tau(\delta)) \cdot \text{vol}(B_\epsilon(x_1) \cup B_\epsilon(x_2)) \\ &= \text{vol}(B_\epsilon(\mathbb{R}^n)) + 2\epsilon \cdot \text{vol}(B_\epsilon(\mathbb{R}^{n-1})) \int_{\pi/3}^{\pi/2} \sin^n(t) dt \\ &= 2\epsilon \cdot \text{vol}(B_\epsilon(\mathbb{R}^{n-1})) \int_0^{\pi/2} \sin^n(t) dt + 2\epsilon \cdot \text{vol}(B_\epsilon(\mathbb{R}^{n-1})) \int_{\pi/3}^{\pi/2} \sin^n(t) dt. \end{aligned}$$

Since $B_\epsilon(f^{-1}(x_1)) \cap B_\epsilon(f^{-1}(x_2)) = \emptyset$, we have

$$\begin{aligned} & (1 + \tau(\delta)) \cdot \text{vol}(B_\epsilon(f^{-1}(x_1)) \cup B_\epsilon(f^{-1}(x_2))) \\ &= 2\text{vol}(B_\epsilon(\mathbb{R}^n)) = 4\epsilon \cdot \text{vol}(B_\epsilon(\mathbb{R}^{n-1})) \int_0^{\pi/2} \sin^n(t) dt. \end{aligned}$$

Note that $f^{-1}(B_\epsilon(x_1) \cup B_\epsilon(x_2)) \supseteq B_\epsilon(f^{-1}(x_1)) \cup B_\epsilon(f^{-1}(x_2))$. Together with that f is volume preserving, we get

$$\begin{aligned} 1 &= \frac{\text{vol}(f^{-1}(B_\epsilon(x_1) \cup B_\epsilon(x_2)))}{\text{vol}(B_\epsilon(x_1) \cup B_\epsilon(x_2))} \geq \frac{\text{vol}(B_\epsilon(f^{-1}(x_1)) \cup B_\epsilon(f^{-1}(x_2)))}{\text{vol}(B_\epsilon(x_1) \cup B_\epsilon(x_2))} \\ &= (1 - \tau(\delta)) \frac{2 \int_0^{\pi/2} \sin^n(t) dt}{\int_0^{\pi/2} \sin^n(t) dt + \int_{\pi/3}^{\pi/2} \sin^n(t) dt}. \end{aligned}$$

This is a contradiction for sufficiently small δ . \square

Let $L(\gamma)$ denote the length of the curve γ . By Lemma 3.2 and 3.3, for any continuous curve $\gamma \subset f(Z^\delta) \subset X^{\tau(\delta)}$, $f^{-1}(\gamma) \subset Z^\delta$ is also a continuous curve with $L(\gamma) \leq L(f^{-1}(\gamma)) \leq 2L(\gamma)$. In particular, $f|_{Z^\delta}$ maps connected components to connected components. We now show that $f|_{Z^\delta}$ is an $\tau(\delta)$ -almost isometry.

Lemma 3.4 (Almost Isometry). *Let $\delta > 0$ be small and $a, b \in Z^\delta$. If $[f(a)f(b)]_X \subset f(Z^\delta)$, then*

$$(3.1) \quad 1 \leq \frac{|ab|_Z}{|f(a)f(b)|_X} \leq 1 + \tau(\delta).$$

Proof. Let $x = f(a), y = f(b) \in f(Z^\delta)$. Note that $[xy]_X \subset f(Z^\delta) \subseteq X^{\tau(\delta)}$, $\gamma = f^{-1}([xy]_X) \subset Z^\delta$ is a Lipschitz curve with $L(f^{-1}([xy]_X)) \leq 2|xy|_X$. Then there is $\rho > 0$, such that $[xy]_X \subset X^{\tau(\delta)}(\rho)$ and $\gamma \subset Z^\delta(\rho)$. Let $\{x_i\}_{i=1}^{N+1}$ be a partition of $[xy]_X$ with $|x_i x_{i+1}| = \epsilon^2$ for all i , where

$\epsilon \ll \min\{\delta\rho, \tau(\delta)\rho\}$. Let $\hat{z}_i = f^{-1}(x_i)$. Consider the union of ϵ -balls of x_i and \hat{z}_i . It's clear that $\bigcup_{i=1}^{N+1} B_\epsilon(x_i)$ satisfies the conditions in the volume formula (Lemma 2.7). Thus we have

$$\begin{aligned}
(1 + \tau(\delta)) \cdot \text{vol} \left(\bigcup_{i=1}^{N+1} B_\epsilon(x_i) \right) \\
&= \text{vol}(B_\epsilon(\mathbb{R}^n)) + \text{vol}(B_\epsilon(\mathbb{R}^{n-1})) \sum_{i=1}^N |x_i x_{i+1}| + O(\epsilon^{n+1}) \sum_{i=1}^N |x_i x_{i+1}| \\
(3.2) \quad &= \text{vol}(B_\epsilon(\mathbb{R}^{n-1})) \sum_{i=1}^N |x_i x_{i+1}| + O(\epsilon^n).
\end{aligned}$$

By Lemma 3.3, it's easy to check that $\bigcup_{i=1}^{N+1} B_\epsilon(\hat{z}_i)$ also satisfies the conditions in the volume formula. Apply the volume formula (2.2) again:

$$(3.3) \quad (1 + \tau(\delta)) \cdot \text{vol} \left(\bigcup_{i=1}^{N+1} B_\epsilon(\hat{z}_i) \right) = \text{vol}(B_\epsilon(\mathbb{R}^{n-1})) \sum_{i=1}^N |\hat{z}_i \hat{z}_{i+1}| + O(\epsilon^n).$$

Using (3.2) and (3.3), together with the fact that f is 1-Lipschitz and volume preserving, we have

$$\begin{aligned}
1 &= \frac{\text{vol} \left(f^{-1} \left(\bigcup_{i=1}^{N+1} B_\epsilon(x_i) \right) \right)}{\text{vol} \left(\bigcup_{i=1}^{N+1} B_\epsilon(x_i) \right)} \geq \frac{\text{vol} \left(\bigcup_{i=1}^{N+1} B_\epsilon(\hat{z}_i) \right)}{\text{vol} \left(\bigcup_{i=1}^{N+1} B_\epsilon(x_i) \right)} \\
&= (1 - \tau(\delta)) \cdot \frac{\text{vol}(B_\epsilon(\mathbb{R}^{n-1})) \sum_{i=1}^N |\hat{z}_i \hat{z}_{i+1}| + O(\epsilon^n)}{\text{vol}(B_\epsilon(\mathbb{R}^{n-1})) \sum_{i=1}^N |x_i x_{i+1}| + O(\epsilon^n)}, \\
&= (1 - \tau(\delta)) \cdot \frac{\sum_{i=1}^N |\hat{z}_i \hat{z}_{i+1}| + O(\epsilon)}{|xy|_X + O(\epsilon)}.
\end{aligned}$$

Let $\epsilon \rightarrow 0$, $N \rightarrow \infty$, we get

$$(1 + \tau(\delta)) \cdot |xy|_X = \lim_{N \rightarrow \infty} \sum_{i=1}^N |\hat{z}_i \hat{z}_{i+1}|_Z \geq L(\gamma) \geq |ab|_Z.$$

□

Let's explain the idea for our next move. Given $\hat{\gamma} \subset Z^\circ$ and $f(\hat{\gamma}) \subset f(Z^\circ)$. It's sufficient to show that $L(f(\hat{\gamma})) \geq L(\hat{\gamma})$. We would like to construct $\sigma_\epsilon \subset f(Z^\delta)$ so that both $\sigma_\epsilon \rightarrow f(\hat{\gamma})$ and $L(\sigma_\epsilon) \rightarrow L(f(\hat{\gamma}))$. Then because $G_Z \subseteq \partial Z$ (Lemma 3.8(i)), $f^{-1}(\sigma_\epsilon) \rightarrow \hat{\gamma}$. Consequently,

$$L(f(\hat{\gamma})) = \lim_{\epsilon \rightarrow 0} L(\sigma_\epsilon) \geq (1 - \tau(\delta)) \liminf_{\epsilon \rightarrow 0} L(f^{-1}(\sigma_\epsilon)) \geq L(\hat{\gamma}).$$

To carry out this idea, the main difficulty is the lack of information about the metric or shape of $f(Z^\delta)$ in $f(Z^\circ)$. Our approach is to find σ_ϵ by modification. First take a partition $\{x_i\}_{i=1}^N$ of $f(\hat{\gamma})$, then perturb the geodesic pieces $[x_i x_{i+1}]_X \subset f(Z^\circ)$ to geodesic pieces $[x'_i x'_{i+1}]_X \subset f(Z^{2\delta})$ with $|x_i x'_i|_X < \epsilon/N$. Then $\sigma_\epsilon = \cup [x'_i x'_{i+1}]_X$ is the desired approximation. The Dimension comparison

Lemma (Lemma 3.7) is crucial to grantee the existence of $[x'_i x'_{i+1}]_X \subset f(Z^{2\delta})$ (Lemma 3.9). For a technical reason, the openness of $f(Z^\circ)$ (Lemma 3.8) is needed as well .

Remark 3.5.

(1.2.1) The above approach essentially relies on the lower curvature bound, because the Dimension comparison Lemma is false without lower curvature bound (Example 1.5).

(1.2.2) The above idea can not be carried on for a path $\hat{\gamma} \subset \partial Z$, because $f^{-1}(\sigma_\epsilon)$ may never converge to the given path $\hat{\gamma}$ for any approximation σ_ϵ of $f(\hat{\gamma})$ (see the following example).

Example 3.6. Consider the gluing of a cylinder $\mathbb{S}(r) \times [0, 1]$ with a disk $\mathbb{D}(r)$ (as the cap). For a gluing path $\hat{\gamma} \subset \mathbb{S}(r) \times \{0\}$, any piece-wise geodesic perturbation of $f(\hat{\gamma})$ in the glued space X “mainly” stays in the disk part, and thus its pre-image will never converge to a path in the the boundary of the cylinder.

The following lemma holds for general $X \in \text{Alex}^n(\kappa)$.

Lemma 3.7 (Dimension comparison). *Let $\Omega_0 \subseteq X \in \text{Alex}^n(\kappa)$ be a subset and $p \in X$ be a fixed point. For each point $x \in \Omega_0$, select one point \bar{x} on a geodesic $[px]_X$. Let Ω be the collection of the \bar{x} s for all $x \in \Omega_0$. If $d_X(p, \Omega) > 0$, then*

$$\dim_H(\Omega) \geq \dim_H(\Omega_0) - 1.$$

Proof. Let $\Gamma = \Omega \times [0, \infty)$, with the metric

$$d((x_1, t_1), (x_2, t_2)) = |x_1 x_2|_X + |t_1 - t_2|,$$

where $x_i \in \Omega$, $t_i \in [0, \infty)$, $i = 1, 2$. Define a map $h : \Omega_0 \rightarrow \Gamma$, $x \mapsto (\bar{x}, |px|_X)$, where $\bar{x} \in [px]_X$ is selected as the above. We claim that the map h is co-Lipschitz, i.e., there is a constant c such that for any $x_1, x_2 \in \Omega_0$,

$$|h(x_1)h(x_2)|_\Gamma \geq c \cdot |x_1 x_2|_X.$$

Then

$$\dim_H(\Omega) + 1 \geq \dim_H(\Gamma) \geq \dim_H(\Omega_0) = n.$$

The above claim is verified by triangle comparison. If geodesics $[px_1]_X$ and $[px_2]_X$ are equivalent (i.e., one lies on the other), then

$$\frac{|h(x_1)h(x_2)|_\Gamma}{|x_1 x_2|_X} = \frac{|\bar{x}_1 \bar{x}_2|_X + ||px_1|_X - |px_2|_X|}{|x_1 x_2|_X} \geq \frac{||px_1|_X - |px_2|_X|}{|x_1 x_2|_X} = 1.$$

Assume that geodesics $[px_1]_X$ and $[px_2]_X$ are not equivalent. Note that $|p\bar{x}_1|_X, |p\bar{x}_2|_X \geq d_X(p, \Omega) > 0$. Thus

$$\frac{|h(x_1)h(x_2)|_\Gamma}{|x_1 x_2|_X} = \frac{|\bar{x}_1 \bar{x}_2|_X + ||px_1|_X - |px_2|_X|}{|x_1 x_2|_X} \geq \frac{|\bar{x}_1 \bar{x}_2|_X}{|x_1 x_2|_X} \geq c(\kappa, d_X(p, \Omega)) > 0.$$

□

Lemma 3.8 (Boundary gluing and dimension). *Assume $G_Z \neq \emptyset$. Let $\hat{p} \in G_Z$.*

- (i) *For $\delta > 0$ small and any $r > 0$, $\dim(B_r(\hat{p}) \setminus Z^\delta) \geq n - 1$. Consequently, $G_Z \subseteq \partial Z$ and thus $f(Z^\circ) = X \setminus f(\partial Z)$ is open.*

- (ii) If in addition the gluing theorem for the spaces of directions (Theorem 1.4) is true, then for any $r > 0$,

$$\dim(B_r(\hat{p}) \cap G_Z) \geq \dim(B_r(f(\hat{p})) \cap G_X) \geq n - 1.$$

Proof. $G_Z \subseteq \partial Z$ is a consequence of (i) due to the fact that the interior non- (n, δ) strained points have dimension at most $n - 2$ (Corollary 2.5). (i) and (ii) share the same proof with a slick modification.

Let $\hat{p} \neq \hat{q} \in G_Z$ with $f(\hat{p}) = f(\hat{q}) = a \in G_X$. Not losing generality, assume $\hat{p} \in Z_1$ and $\hat{q} \in Z_\alpha$ (α may equal 1). For both (i) and (ii), because f is 1-Lipschitz and $\dim_H \leq \dim_T$, it's sufficient to consider the Hausdorff dimension for $f(B_r(\hat{p}) \setminus Z^\delta)$ and $f(B_r(\hat{p}) \cap G_Z) = f(B_r(\hat{p})) \cap G_X$ respectively. Note that $f(B_r(\hat{p}) \setminus Z^\delta) = f(B_r(\hat{p})) \setminus f(Z^\delta)$ by Lemma 3.2.

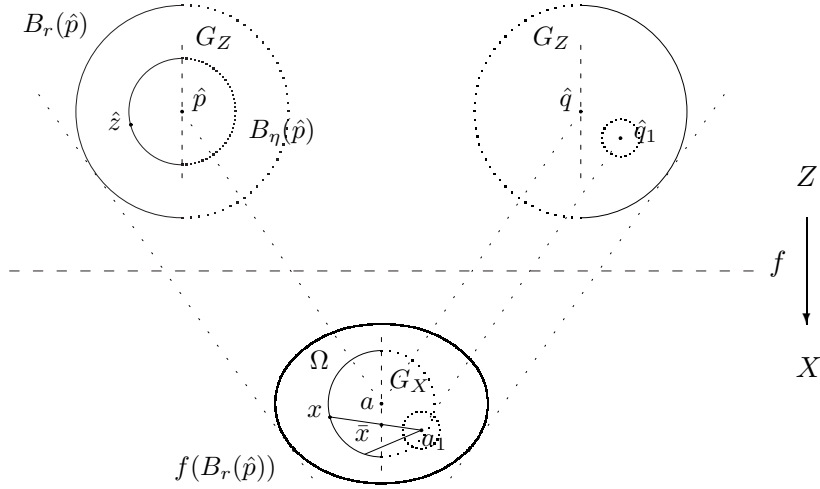


Figure 2

By Lemma 3.2, select $\delta > 0$ small so that $f(Z^\delta) \cap G_X = \emptyset$. By Theorem 2.6, for any $\eta > 0$ small, there is $\hat{q}_1 \in Z_\alpha^{\text{Reg}}$ with $|\hat{q}\hat{q}_1|_{Z_\alpha} < \eta$. By Lemma 3.1, $a_1 = f(\hat{q}_1) \in X^{\text{Reg}}$. Let $\Omega = f(B_\eta(\hat{p}) \cap Z_1^{\delta/2})$. By the volume preserving, it's clear that

$$\dim_H(\Omega) = n.$$

We first claim that for any $x \in \Omega$, $[xa_1]_X - f(Z^\delta) \neq \emptyset$. If not so, then $[xa_1]_X \subset f(Z^\delta)$. Let $\hat{z} = f^{-1}(x)$. By the almost isometry of f (Lemma 3.4), we get $|xa_1|_X = (1 - \tau(\delta))|\hat{z}\hat{q}_1|_Z$. Consequently,

$$\begin{aligned} 2\eta &\geq |\hat{z}\hat{p}|_Z + |\hat{q}\hat{q}_1|_Z \geq |xa|_X + |aa_1|_X \\ &\geq |xa_1|_X = (1 - \tau(\delta))|\hat{z}\hat{q}_1|_Z \geq (1 - \tau(\delta))(|\hat{p}\hat{q}|_Z - 2\eta), \end{aligned}$$

which yields a contradiction by choosing δ and η small.

Take $\bar{x} \in [xa_1]_X \setminus f(Z^\delta)$ which is closest to x (see Figure 2). It's clear that $\bar{x} \neq x$. Moreover, $\bar{x} \notin f(Z^\delta)$ because $f(Z^\delta) = X \setminus f(Z \setminus Z^\delta)$ is open in X (by Lemma 3.2) and thus $[xa_1]_X \setminus f(Z^\delta)$ is closed. We claim that $\bar{x} \in f(B_r(\hat{p}))$ and furthermore $\bar{x} \in f(B_r(\hat{p})) \cap G_X$ if Theorem 1.4 is true.

Assuming the claim, let Ω_1 be the collection of all \bar{x} for all $x \in \Omega$. Then $\Omega_1 \subset f(B_r(\hat{p})) \setminus f(Z^\delta)$ and $\Omega_1 \subset f(B_r(\hat{p})) \cap G_X$ under the assumptions as in Lemma 3.8(ii).

Note that $a_1 \in X^{\text{Reg}}$. There is a small ball $B_\epsilon(a_1) \subset f(Z^{\delta/2})$. Thus for any of the above selected $\bar{x} \notin f(Z^\delta)$, we have $|\bar{x}a_1|_X \geq \epsilon$. By the Dimension comparison Lemma (Lemma 3.7), we have

$$\dim_H(\Omega_1) \geq \dim_H(\Omega) - 1 = n - 1.$$

It remains to check the claim. We shall find $\bar{z} \in B_r(\hat{p})$ such that $\bar{x} = f(\bar{z})$ and show that $\bar{z} \in G_Z$ if Theorem 1.4 is satisfied. By the construction, $[x\bar{x}]_X \subset f(Z^\delta)$. Let $x_i \in [x\bar{x}]_X$ with $x_i \rightarrow \bar{x}$. By the almost isometry of f (Lemma 3.4), there are $\hat{z}_i = f^{-1}(x_i) \in Z^\delta$ such that

$$(3.4) \quad (1 + \tau(\delta))|xx_i|_X = |\hat{z}\hat{z}_i|_Z.$$

Passing to a subsequence, let $\bar{z} = \lim_{i \rightarrow \infty} \hat{z}_i$. Clearly, because f is continuous, we have $f(\bar{z}) = \bar{x}$ and

$$(3.5) \quad |\hat{z}\bar{z}|_Z = \lim_{i \rightarrow \infty} |\hat{z}\hat{z}_i|_Z = (1 + \tau(\delta)) \lim_{i \rightarrow \infty} |xx_i|_X = (1 + \tau(\delta))|x\bar{x}|_X.$$

Thus

$$\begin{aligned} |\hat{p}\bar{z}|_Z &\leq |\hat{p}\hat{z}|_Z + |\hat{z}\bar{z}|_Z \\ &\leq \eta + (1 + \tau(\delta))|x\bar{x}|_X \leq \eta + (1 + \tau(\delta))|xa_1|_X \leq (3 + \tau(\delta))\eta. \end{aligned}$$

Choosing $\eta > 0$ small, we will have $\bar{z} \in B_r(\hat{p})$. Because $a_1 \in X^{\text{Reg}}$ and by Theorem 5.1 and 5.2, we see that $[xa_1]_X \subset X^{\text{Reg}}$. In particular, $\bar{x} \in X^{\text{Reg}}$.

If assume Theorem 1.4 but $\bar{x} \notin G_X$, then

$$\text{vol}(\Sigma_{\bar{z}}) = \text{vol}(\Sigma_{\bar{x}}) = \text{vol}(\mathbb{S}_1^{n-1}),$$

Consequently, $\bar{z} \in Z^{\text{Reg}}$. This contradicts to the selection that $\bar{x} \notin f(Z^\delta)$. \square

Our plan is to use Lemma 3.8(i) and Lemma 3.7 to construct the desired perturbation of $f(\hat{\gamma})$. This will enable us to prove that $f|_{Z^\circ}$ is an Internal isometry (Lemma 1.6). Using this and induction on Theorem 1.1, we establish Theorem 1.4, and then Lemma 3.8(ii) follows. Together with the fact $G_Z \subseteq \partial Z$, Lemma 1.7 is proved.

Lemma 3.9 (The single perturbation). *For any $p \in f(Z^\delta)$ and $x \in X$. If $[px]_X \subset U \subseteq f(Z_\alpha^\circ)$, where U is a convex neighborhood of x in X , then for any $\epsilon > 0$, there is $x' \in B_\epsilon(x)$ such that $[px']_X \subset f(Z^{2\delta})$.*

Proof. If the assertion is not true, then for any $x' \in B_\epsilon(x)$, $[px']_X \setminus f(Z^{2\delta})$ contains at least one point. Let $\Omega = \{y' \in [px']_X \setminus f(Z^{2\delta}) : x' \in B_\epsilon(x)\}$. Because $p \in f(Z^\delta)$, there is a small ball such that $B_r(p) \subset f(Z^{2\delta})$ and thus $d_X(p, \Omega) \geq r$. By the Dimension comparison Lemma, we get

$$(3.6) \quad \dim_H(\Omega) \geq n - 1.$$

On the other hand, take $\epsilon > 0$ small so that $B_\epsilon(q) \subset U$. Because U is convex in X , we have $[px']_X \subset U \subseteq f(Z_\alpha^\circ)$ for all $x' \in B_\epsilon(x)$. Consequently, $\Omega \subset f(Z_\alpha^\circ) \setminus f(Z^{2\delta}) = f(Z_\alpha^\circ \setminus Z^{2\delta})$ by Lemma 3.2. Thus

$$\dim_H(\Omega) \leq \dim_H(f(Z^\circ \setminus Z^{2\delta})) \leq \dim_H(Z^\circ \setminus Z^{2\delta}) \leq n - 2,$$

which contradicts to the dimension estimate (3.6). \square

Proof of Lemma 1.6. Let $\hat{\gamma} : [0, 1] \rightarrow Z^\circ$ be a Lipschitz curve and $\gamma = f(\hat{\gamma})$. Clearly, γ is a Lipschitz curve since f is 1-Lipschitz. It remains to show $L(\gamma) \geq L(\hat{\gamma})$. Note that by Lemma 3.8(i), $f(Z^\circ)$ is open. Then for each $x \in \gamma$, there is a convex neighborhood $U_x \subset f(Z^\circ)$. The existence of such convex neighborhood is referred to [PP] 4.3. Because γ is compact in X , there is a finite covering $\{U_{x_{2i}}\}_{i=0}^N$. Let $t_{2i} \geq 0$ so that $\gamma(t_{2i}) = x_{2i}$. Choose the corresponding covering from the interval $[0, 1]$ so that $0 = t_0 < t_2 < \dots < t_{2N} = 1$, $\gamma(t_0) = x_0$, $\gamma(t_{2N}) = x_{2N}$ and $\gamma \cap U_{x_{2i}} \cap U_{x_{2(i+1)}} \neq \emptyset$. Let $x_{2i+1} \in \gamma \cap U_{x_{2i}} \cap U_{x_{2(i+1)}}$, for $i = 0, 1, \dots, N-1$. Then $0 = t_0 < t_1 < t_2 < \dots < t_{2N} = 1$ and we have

$$L(\gamma) \geq \sum_{j=0}^{2N-1} |x_j x_{j+1}|_X.$$

Now we use Lemma 3.9 to find the right perturbation of $\cup [x_j x_{j+1}]_X$. First choose $x'_0 \in f(Z^{\delta/2^{2N}}) \cap B_{\epsilon/2N}(x_0) \cap U_{x_0}$. By the convexity of U_{x_0} , we have $[x'_0 x_1]_X \subset U_{x_0} \subset f(Z^\circ)$. By Lemma 3.9, there is $x'_1 \in B_{\epsilon/2N}(x_1) \cap U_{x_0} \cap U_{x_2}$ such that $[x'_0 x'_1]_X \subset f(Z^{\delta/2^{2N-1}})$. By the convexity of U_{x_2} , we have $[x'_1 x_2]_X \subset U_{x_2} \subset f(Z^\circ)$. Proceeding the above adjustment recursively for $j = 1, 2, \dots, 2N$, we get a sequence $\{x'_j\}_{j=0}^{2N}$ with $x'_j \in B_{\epsilon/2N}(x_j)$ such that $[x'_j x'_{j+1}]_X \subset f(Z^{\delta/2^{2N-(j+1)}}) \subset f(Z^\delta)$ for each j . Because x'_j are $\epsilon/2N$ -close to x_j , we have

$$L(\gamma) \geq \sum_{j=0}^{2N-1} |x_j x_{j+1}|_X \geq \sum_{j=0}^{2N-1} \left(|x'_j x'_{j+1}|_X - \frac{2\epsilon}{2N} \right) = \sum_{j=0}^{2N-1} |x'_j x'_{j+1}|_X - 2\epsilon.$$

Let $\hat{z}_i = f^{-1}(x'_i)$. By the almost isometry (Lemma 3.4), we have $|x'_i x'_{i+1}|_X = (1 - \tau(\delta)) |\hat{z}_i \hat{z}_{i+1}|_Z$. Note that $\bigcup_{i=0}^{2N-1} [\hat{z}_i \hat{z}_{i+1}]_Z \rightarrow \hat{\gamma}$ as $\epsilon \rightarrow 0$, since $G_Z \cap Z^\circ = \emptyset$. Therefore, letting $\epsilon \rightarrow 0$, we have

$$L(\gamma) = \lim_{\epsilon \rightarrow 0} \sum_{i=0}^{2N-1} |x'_i x'_{i+1}|_X = (1 - \tau(\delta)) \lim_{\epsilon \rightarrow 0} \sum_{i=0}^{2N-1} |\hat{z}_i \hat{z}_{i+1}|_Z \geq (1 - \tau(\delta)) L(\hat{\gamma}).$$

Let $\delta \rightarrow 0$, we will get the desired result. \square

Corollary 3.10. *Let $\hat{z} \in Z$ and $\gamma : [0, T] \rightarrow Z$ be a quasi-geodesic with $\gamma(0) = \hat{z}$ and $\gamma((0, T]) \subset Z^\circ$. Then $f(\gamma)$ is a quasi-geodesic in X .*

Proof. Let $t_i > 0$ and $t_i \rightarrow 0$. Note that $f(Z^\circ)$ is open, thus for each $x \in \gamma((t_i, T])$, there is a convex neighborhood $U_x \subset f(Z^\circ)$. Consequently, $f(\gamma((t_i, T]))$ is quasi-geodesic in X , since $f|_{Z^\circ}$ is an isometry. Because the limit of quasi-geodesics is a quasi-geodesic, we get that $f(\gamma) = \lim_{i \rightarrow \infty} f(\gamma((t_i, T]))$ is a quasi-geodesic. \square

We prove Theorem 1.1 by induction on dimensions. By the inductive hypothesis, the following lemma implies that the space of directions of the boundary points are also glued as described in Theorem 1.1, i.e., Theorem 1.4 holds. Recall that $\Sigma_{f^{-1}(x)} = \coprod_{\beta} \Sigma_{\hat{z}_\beta}$, where $x \in X$ and $\{\hat{z}_\beta\} = f^{-1}(x) \subset Z$.

Lemma 3.11. *For any $x \in X$, the following holds:*

- (i) $\text{vol}(\Sigma_{f^{-1}(x)}) = \text{vol}(\Sigma_x)$,
- (ii) $f^{-1}(x)$ is finite,
- (iii) f induces a shrinking map $df_x : \Sigma_{f^{-1}(x)} \rightarrow \Sigma_x$.

Proof. Let $f^{-1}(x) = \{\hat{z}_\beta\}$. We first define a 1-Lipschitz map $df_x : \Sigma_{\hat{z}_\beta} \rightarrow \Sigma_x$ over $\Sigma_{\hat{z}_\beta}$ for each β . Assume $\hat{z}_\beta \in Z_{\alpha_\beta}$. For any $\xi \in \Sigma_{\hat{z}_\beta}^\circ$, let $\gamma : [0, T] \rightarrow Z_{\alpha_\beta}$ be a quasi-geodesic with $\gamma(0) = \hat{z}_\beta$, $\gamma^+(0) = \xi$ and $\gamma((0, T]) \subset Z_{\alpha_\beta}^\circ$. Let $\sigma(t) = f(\gamma(t))$. By Corollary 3.10, σ is a quasi-geodesic in X . We define $df_x(\xi) = \sigma^+(0)$. Note that $f|_{Z_{\alpha_\beta}^\circ}$ is an isometry, thus $df_x|_{\Sigma_{\hat{z}_\beta}^\circ}$ is also an isometry, since the intrinsic distance in $df_x(\Sigma_{\hat{z}_\beta}^\circ)$ is determined by the limit of comparison angles whose opposite sides are taken as geodesics in $f(Z_{\alpha_\beta}^\circ)$. Because $\Sigma_{\hat{z}_\beta}$ is compact and $\Sigma_{\hat{z}_\beta}^\circ$ is dense in $\Sigma_{\hat{z}_\beta}$, df_x can be uniquely extended to a 1-Lipschitz map over $\Sigma_{\hat{z}_\beta}$. Consequently, we get a 1-Lipschitz map

$$df_x : \Sigma_{df^{-1}(x)} = \coprod_{\beta} \Sigma_{\hat{z}_\beta} \rightarrow \Sigma_x.$$

To show (i), we first observe that $df_x(\Sigma_{\hat{z}_\beta}^\circ) \cap df_x(\partial \Sigma_{\hat{z}_\beta}) = \emptyset$. If not so, assume $df_x(a) = df_x(b)$, where $a \in \Sigma_{\hat{z}_\beta}^\circ$ and $b \in \partial \Sigma_{\hat{z}_\beta}$. Take $b' \in [ab]_{\Sigma_{\hat{z}_\beta}}$ with $|b'b|_{\Sigma_{\hat{z}_\beta}} = \frac{1}{4}|ab|_{\Sigma_{\hat{z}_\beta}}$. Then $b' \in \Sigma_{\hat{z}_\beta}^\circ$. Because $df_x|_{\Sigma_{\hat{z}_\beta}^\circ}$ is an isometry,

$$\begin{aligned} \frac{3}{4}|ab|_{\Sigma_{\hat{z}_\beta}} &= |ab'|_{\Sigma_{\hat{z}_\beta}} = |df_x(a)df_x(b')|_{\Sigma_x} \\ &= |df_x(b)df_x(b')|_{\Sigma_x} \leq |bb'|_{\Sigma_{\hat{z}_\beta}} = \frac{1}{4}|ab|_{\Sigma_{\hat{z}_\beta}}, \end{aligned}$$

which is a contradiction. To get (3.11.1), it's sufficient to show that $df_x(\Sigma_{\hat{z}_i}^\circ) \cap df_x(\Sigma_{\hat{z}_j}^\circ) = \emptyset$ for $i \neq j$. Let $a \in \Sigma_{\hat{z}_i}^\circ$ and $b \in \Sigma_{\hat{z}_j}^\circ$, where $\hat{z}_i \in Z_{\alpha_i}$, $\hat{z}_j \in Z_{\beta_j}$. Let γ, σ be geodesics in X from x whose directions are ϵ -close to $df_x(a), df_x(b)$ and $\gamma((0, T]) \subset Z_{\alpha_i}^\circ$, $\sigma((0, T]) \subset Z_{\beta_j}^\circ$. For each $t, s \in (0, T]$, take $u \in [\gamma(t)\sigma(s)]_X \cap f(\partial Z_{\alpha_i})$ which is closest to $\gamma(t)$. Clearly $u \in f(\partial Z_{\alpha_i})$ and $u \neq \gamma(t)$. Passing to a subsequence, we get $\xi = \lim_{t,s \rightarrow 0} \uparrow_x^u \in df_x(\partial \Sigma_{\hat{z}_j})$. Then

$$\begin{aligned} |df_x(a)df_x(b)|_{\Sigma_x} &\geq \lim_{s,t \rightarrow 0} \tilde{\angle}_\kappa \left(x_{\sigma(s)}^{\gamma(t)} \right) - 2\epsilon \\ &\geq \lim_{s,t \rightarrow 0} \tilde{\angle}_\kappa \left(x_u^{\gamma(t)} \right) - 2\epsilon = |df_x(a)\xi|_{\Sigma_x} - 2\epsilon > 0. \end{aligned}$$

The last inequality holds for ϵ small since $df_x(\Sigma_{\hat{z}_j}^\circ) \cap df_x(\partial \Sigma_{\hat{z}_j}) = \emptyset$.

(ii) follows by the proof of Theorem 1.1(iii) since it only requires the volume equation

$$\text{vol}(\Sigma_x) = \text{vol} \left(\coprod_{\beta} \Sigma_{\hat{z}_\beta} \right).$$

(iii) It remains to show that df_x is onto. This follows by the fact that df_x is continuous, $df_x(\coprod_{\beta} \Sigma_{\hat{z}_\beta})$ is dense and $\coprod_{\beta} \Sigma_{\hat{z}_\beta}$ is a union of finitely many compact spaces. \square

4. LENGTH PRESERVING AND SHRINKING RIGIDITY THEOREM

The main effort for this section is to show that f preserves the length of path (Lemma 1.8). This is not true without assuming a lower curvature bound (Example 1.5). The key lemma (Lemma 4.2) relies on the local structures near the gluing/glued points (Lemma 4.1) and the

first variation formula. If not stated otherwise, the assumptions for the lemmas in this section will be the same as in Theorem 1.1.

Lemma 4.1 (Locally almost conic gluing). *Let $df_x : \Sigma_{\hat{z}_\beta} \rightarrow \Sigma_x$ be defined as in Lemma 3.11. For any $\delta > 0$ small and $\hat{a} \in \partial Z_\alpha$, $a = f(\hat{a})$, there is a neighborhood $U_{\hat{a}}$ of \hat{a} in Z_α such that for any $\hat{q} \in U_{\hat{a}} \cap \partial Z_\alpha$ and $q = f(\hat{q})$,*

$$|df_x(\uparrow_{\hat{q}}^{\hat{a}}) \uparrow_q^a|_{\Sigma_q} < 2\delta.$$

Proof. Let $T_q^{\hat{q}}(\partial Z_\alpha) = df_q(\partial \Sigma_{\hat{q}}) = \left\{ \lim_{\hat{x} \rightarrow \hat{q}} \uparrow_q^x : x = f(\hat{x}), \hat{x} \in \partial Z_\alpha \right\}$ denote the tangent space of $f(\partial Z_\alpha)$ at q respect to \hat{q} . We first show that there is $v \in T_q^{\hat{q}}(\partial Z_\alpha)$ so that

$$(4.1) \quad |\uparrow_q^a v|_{\Sigma_q} > \pi - \delta.$$

If this is not true, then there is a sequence $\hat{q}_i \rightarrow \hat{a}$ and $q_i = f(\hat{q}_i) \rightarrow a$ such that

$$|\uparrow_{q_i}^a v|_{\Sigma_{q_i}} \leq \pi - \delta$$

for all $v \in T_q^{\hat{q}}(\partial Z_\alpha)$. Let (X_i, a) be the rescaled space of (X, a) by $1/|\hat{a}\hat{q}_i|_{Z_\alpha} \rightarrow \infty$. Then $(X_i, a) \xrightarrow{d_{GH}} (C_0(\Sigma_a), \bar{a})$, where $C_0(\Sigma_a)$ is the tangent cone of a and \bar{a} is the cone vertex. Also, we have $(Z_\alpha, \hat{a}) \xrightarrow{d_{GH}} (C_0(\Sigma_{\hat{a}}), \hat{\bar{a}})$. Let $\bar{q} \in C_0(\Sigma_a)$ be the limit of q_i . Passing to a subsequence, let $\hat{\bar{q}} \in C_0(\Sigma_{\hat{a}})$ be the corresponding limit of \hat{q}_i in the rescaling. Let $\bar{f} : C_0(\Sigma_{f^{-1}(a)}) \rightarrow C_0(\Sigma_a)$ be the limit projection map. By Theorem 1.4, $C_0(\Sigma_a)$ is a glued space of $C_0(\Sigma_{f^{-1}(a)})$ along their boundaries $\partial \Sigma_{f^{-1}(a)} \times \mathbb{R}$. Moreover, we have that

- (i) there is a $1 + \tau(1/i)$ -Lipschitz onto map from the sequence $T_{q_i}^{\hat{q}_i}(\partial Z_\alpha)$ to $T_{\bar{q}}^{\hat{\bar{q}}}(\partial Z_\alpha)$ for i large;
- (ii) $T_{\bar{q}}^{\hat{\bar{q}}}(\partial Z_\alpha) = df_{\bar{q}}(\partial \Sigma_{\hat{\bar{q}}})$;
- (iii) for $u, v \in \partial \Sigma_{f^{-1}(a)}$ with $\xi = df_a(u) = df_a(v) \in \Sigma_a$, we have that $\bar{f}(u \times \mathbb{R}) = \bar{f}(v \times \mathbb{R}) = \xi \times \mathbb{R}$ is a geodesic.

By (i), we have

$$(4.2) \quad |\uparrow_{\bar{q}}^{\bar{a}} v|_{\Sigma_{\bar{q}}} \leq \pi - \delta,$$

for all $v \in T_{\bar{q}}^{\hat{\bar{q}}}(\partial Z_\alpha)$. Consider the geodesic $[\hat{\bar{q}}\hat{\bar{a}}]_{C_0(\Sigma_{\hat{a}})} \subset \uparrow_{\hat{\bar{a}}}^{\hat{\bar{q}}} \times \mathbb{R}$ and extend it to \hat{q}_1 . Let $\bar{q}_1 = \bar{f}(\hat{q}_1)$. By property (iii), $[\bar{q}_1 \bar{a}]_{C_0(\Sigma_a)} = \bar{f}([\hat{\bar{q}}_1 \hat{\bar{a}}_1]_{C_0(\Sigma_{\hat{a}})})$ is a geodesic in $C_0(\Sigma_a)$ connecting \bar{q}_1 and \bar{a} and passing through \bar{q} . Note that $\uparrow_{\bar{q}}^{\bar{q}_1} = df_{\bar{q}}(\uparrow_{\hat{\bar{q}}}^{\hat{\bar{q}}_1}) \in df_{\bar{q}}(\partial \Sigma_{\hat{\bar{q}}}) = T_{\bar{q}}^{\hat{\bar{q}}}(\partial Z_\alpha)$ by (ii). This contradicts to (4.2) since $|\uparrow_{\bar{q}}^{\bar{a}} \uparrow_{\bar{q}}^{\bar{q}_1}|_{\Sigma_{\bar{q}}} = \pi$.

Note that $f([\hat{q}\hat{a}])$ is a quasi-geodesic in X jointing q and a . For the above selected v , by a similar argument, we see that

$$(4.3) \quad |df_x(\uparrow_{\hat{q}}^{\hat{a}}) v|_{\Sigma_q} > \pi - \delta.$$

Because $\Sigma_q \in \text{Alex}^{n-1}(i)$, we have

$$|\uparrow_q^a v|_{\Sigma_q} + |df_x(\uparrow_{\hat{q}}^{\hat{a}}) v|_{\Sigma_q} + |df_x(\uparrow_{\hat{q}}^{\hat{a}}) \uparrow_q^a|_{\Sigma_q} \leq 2\pi.$$

Together with (4.1) and (4.3), we get the desired inequality. \square

Using the above lemma the the first variation, we are able to show that f almost preserves the length of path locally.

Lemma 4.2. *Let $\delta > 0$ be small. Then for any $\hat{a} \in \partial Z_\alpha$, there is $r_0 = r_0(\hat{a}) > 0$ such that for any $\hat{b} \in B_{r_0}(\hat{a})$,*

$$1 \geq \frac{|f(\hat{a})f(\hat{b})|_X}{|\hat{a}\hat{b}|_{Z_\alpha}} \geq 1 - \delta.$$

Proof. Let $\hat{U} \subset Z_\alpha$ be the neighborhood of \hat{a} chosen in Lemma 4.1 and

$$r_0 = \frac{1}{10} \sup\{r : B_r(\hat{a}) \subset \hat{U}\} > 0.$$

Let $a = f(\hat{a})$. Given $\hat{b} \in B_{r_0}(\hat{a})$, and $b = f(\hat{b})$. It's sufficient to find a path $\hat{\gamma}$ from \hat{b} to \hat{a} in Z_α such that $(1 - \delta)L(\hat{\gamma}) \leq |ba|_X$. Let

$$s = \inf\{|f(\hat{\gamma}(T))a|_X : \text{there is a path } \hat{\gamma} : [0, T] \rightarrow Z_\alpha \text{ with } \hat{\gamma}(0) = \hat{b}, \\ \text{and satisfies } |ba|_X - |f(\hat{\gamma}(T))a|_X \geq (1 - \delta)L(\hat{\gamma})\}.$$

Clearly $|ba|_X \geq s \geq 0$. We first show that $s = 0$. Assume $s > 0$, then there is a path $\hat{\gamma} \subset B_{r_0}(\hat{a}) \subset Z_\alpha$ such that $\hat{\gamma}(0) = \hat{b}$, $\hat{\gamma}(T) = \hat{q}$, $q = f(\hat{q})$, $s = |qa|_X > 0$, and

$$(4.4) \quad |ba|_X - |qa|_X \geq (1 - \delta)L(\hat{\gamma}).$$

Starting from \hat{q} , we will find an extension of $\hat{\gamma}$ toward \hat{a} which also satisfies (4.4). By a little perturbation if $\uparrow_{\hat{q}}^{\hat{a}} \notin \Sigma_{\hat{q}}^\circ$, take $\hat{\xi} \in \Sigma_{\hat{q}}^\circ$, such that

$$(4.5) \quad |\hat{\xi} \uparrow_{\hat{q}}^{\hat{a}}|_{\Sigma_{\hat{q}}} < \delta.$$

Take a quasi-geodesic $\hat{\sigma} : [0, \epsilon] \rightarrow Z_\alpha$ with $\hat{\sigma}(0) = \hat{q}$, $\hat{\sigma}^+(0) = \hat{\xi}$ and $\hat{\sigma}((0, \epsilon]) \subset Z_\alpha^\circ$. Let $\hat{u} = \hat{\sigma}(\epsilon) \in Z_\alpha^\circ$, $u = f(\hat{u}) \in f(Z_\alpha^\circ)$. Join u and a by a geodesic $[ua]_X$ in X . Let $q_1 \in [ua]_X \cap f(\partial Z_\alpha)$ which is closest to u . Because $f^{-1}(u) = \hat{u} \in Z_\alpha^\circ$, there is $\hat{q}_1 \in \partial Z_\alpha$ such that $q_1 = f(\hat{q}_1)$, $[uq_1]_{Z_\alpha} \subset Z_\alpha^\circ$, $[uq_1]_X = f([\hat{u}\hat{q}_1]_{Z_\alpha})$ and $|\hat{u}\hat{q}_1|_{Z_\alpha} = |uq_1|_X$. We claim that

$$(4.6) \quad |qa|_X - |q_1a|_X \geq (1 - \delta)(\epsilon + |uq_1|_X) = (1 - \delta)(L(\hat{\sigma}) + |\hat{u}\hat{q}_1|_{Z_\alpha}).$$

Then $|q_1a|_X < |qa|_X = s$. Summing (4.6) with (4.4), we get

$$(4.7) \quad |ba|_X - |q_1a|_X \geq (1 - \delta)(L(\hat{\gamma}) + L(\hat{\sigma}) + |\hat{u}\hat{q}_1|_{Z_\alpha}) \\ = (1 - \delta)L(\hat{\gamma} \cup \hat{\sigma} \cup [\hat{u}\hat{q}_1]_{Z_\alpha}),$$

where $\hat{\gamma} \cup \hat{\sigma} \cup [\hat{u}\hat{q}_1]_{Z_\alpha} : \hat{b} \rightarrow \hat{q} \rightarrow \hat{u} \rightarrow \hat{q}_1$ is a continuous path. This contradicts to the assumption that $s = |qa|_X$ is the infimum.

To see (4.6), consider the triangle $\triangle uqa \subset X$ which consists of $[qa]_X$, $[ua]_X$ and quasi-geodesic $\sigma = f(\hat{\sigma})$ (by Corollary 3.10). Note that for $\delta > 0$ small and any path $\hat{\gamma}$ satisfying (4.4), we have

$$(4.8) \quad |\hat{q}\hat{a}|_{Z_\alpha} \leq |\hat{b}\hat{q}|_{Z_\alpha} + |\hat{b}\hat{a}|_{Z_\alpha} \leq L(\hat{\gamma}) + r_0 \\ \leq 2(|ba|_X - |qa|_X) + r_0 < 2|\hat{b}\hat{a}|_{Z_\alpha} + r_0 \leq 3r_0.$$

Thus $\hat{q} \in B_{4r_0}(\hat{a}) \subset \hat{U}$. It's clear that $\sigma^+(0) = df_x(\hat{\xi})$. By Lemma 4.1,

$$(4.9) \quad \begin{aligned} |\sigma^+(0) \uparrow_q^a|_{\Sigma_q} &\leq |\sigma^+(0) df_x(\uparrow_{\hat{q}}^{\hat{a}})|_{\Sigma_q} + |df_x(\uparrow_{\hat{q}}^{\hat{a}}) \uparrow_q^a|_{\Sigma_q} \\ &\leq |\hat{\xi} \uparrow_{\hat{q}}^{\hat{a}}|_{\Sigma_{\hat{q}}} + |df_x(\uparrow_{\hat{q}}^{\hat{a}}) \uparrow_q^a|_{\Sigma_q} < 3\delta. \end{aligned}$$

Note that $L(\sigma) = L(\hat{\sigma}) = \epsilon$. By the first variation formula,

$$|uq_1|_X + |q_1a|_X = |ua|_X \leq |qa|_X - \cos(3\delta)\epsilon + o(\epsilon).$$

Take $\epsilon > 0$ small so that $o(\epsilon) < \frac{1}{2}\delta\epsilon$. Then for $\delta > 0$ small,

$$\begin{aligned} |qa|_X - |q_1a|_X &\geq |uq_1|_X + \cos(3\delta)\epsilon - \frac{1}{2}\delta\epsilon \\ &\geq |uq_1|_X + (1 - \delta)\epsilon \geq (1 - \delta)(|uq_1|_X + \epsilon). \end{aligned}$$

Since $s = 0$, let $\hat{\gamma} : [0, T] \rightarrow Z_\alpha$ be a path from \hat{b} satisfying

$$|ba|_X \geq (1 - \delta)L(\hat{\gamma}) \quad \text{and} \quad f(\hat{\gamma}(T)) = a.$$

It remains to show that $\hat{\gamma}(T) = \hat{a}$. Since $f^{-1}(a)$ is finite (Theorem 1.1(iii)), we can take \hat{U} small enough so that $f^{-1}(a) \cap \hat{U} = \{\hat{a}\}$. Thus it's sufficient to check $\hat{\gamma}(T) \in \hat{U}$. By a similar estimation as (4.8), we get that

$$\begin{aligned} |\hat{\gamma}(T)\hat{a}|_{Z_\alpha} &\leq |\hat{b}\hat{\gamma}(T)|_{Z_\alpha} + |\hat{b}\hat{a}|_{Z_\alpha} \leq L(\hat{\gamma}) + |\hat{b}\hat{a}|_{Z_\alpha} \\ &\leq 2|ba|_X + |\hat{b}\hat{a}|_{Z_\alpha} \leq 3r_0. \end{aligned}$$

Thus $\hat{\gamma}(T) \in B_{4r_0}(\hat{a}) \subset \hat{U}$. □

In the following we give a proof of the length preserving.

Proof of the length preserving (Lemma 1.8). By Lemma 1.7, it remains to show that $L(f(\hat{\gamma})) \geq L(\hat{\gamma})$ for any Lipschitz path $\hat{\gamma} : [0, 1] \rightarrow \partial Z_\alpha$. Let $\delta > 0$ be small. For each $\hat{x} \in \hat{\gamma}$, there is an open ball $B(\hat{x})$ satisfies Lemma 4.2. Since $\hat{\gamma}$ is compact, there is a finite covering $\{B(\hat{x}_{2i})\}_{i=0}^N$. Let $t_{2i} \geq 0$ so that $\hat{\gamma}(t_{2i}) = \hat{x}_{2i}$. Choose the covering so that $0 = t_0 < t_2 < \dots < t_{2N} = 1$, $\hat{\gamma}(0) = \hat{x}_0$, $\hat{\gamma}(1) = \hat{x}_{2N}$ and $\hat{\gamma} \cap B(\hat{x}_{2i}) \cap B(\hat{x}_{2(i+1)}) \neq \emptyset$. Let $\hat{x}_{2i+1} \in \hat{\gamma} \cap B(\hat{x}_{2i}) \cap B(\hat{x}_{2(i+1)})$, for $i = 0, 1, \dots, N-1$. Then $0 = t_0 < t_1 < t_2 < \dots < t_{2N} = 1$. For any $\epsilon > 0$, by choosing the size of $U_{\hat{x}}$ small, we have

$$\sum_{j=0}^{2N} |\hat{x}_j \hat{x}_{j+1}|_{Z_\alpha} \geq L(\hat{\gamma}) - \epsilon.$$

Let $x_j = f(\hat{x}_j) \in f(\hat{\gamma})$. By Lemma 4.2,

$$\begin{aligned} L(f(\hat{\gamma})) &\geq \sum_{j=0}^{2N} |x_j x_{j+1}|_X \geq (1 - \delta) \sum_{j=0}^{2N} |\hat{x}_j \hat{x}_{j+1}|_{Z_\alpha} \\ &\geq (1 - \delta)(L(\hat{\gamma}) - \epsilon). \end{aligned}$$

Let $\epsilon, \delta \rightarrow 0$, we get $L(f(\hat{\gamma})) \geq L(\hat{\gamma})$. □

By the Shrinking rigidity Theorem and the the gluing of spaces of directions (Theorem 1.4), we have the following properties for the gluing points.

Proposition 4.3.

- (i) $f(Z^\delta) \subseteq X^\delta \setminus G_X \subseteq f(Z^{\tau(\delta)})$ for $\delta > 0$ small. In particular, $X^{Reg} \setminus G_X = f(Z^{Reg})$.
- (ii) $\partial X \subseteq f(\partial Z)$.

Proof. (i) $f(Z^\delta) \subseteq X^\delta \setminus G_X$ is clear by the Theorem 1.1. For any $x \in X^\delta \setminus G_X$, let $\hat{z} = f^{-1}(x)$. By Theorem 1.4,

$$\text{vol}(\Sigma_{\hat{z}}) = \text{vol}(\Sigma_x) \geq \text{vol}(\mathbb{S}_1^{n-1}) - \tau(\delta).$$

By Almost Maximum Volume Theorem (Theorem 2.8), $\hat{z} \in Z^{\tau(\delta)}$.

(ii) It's equivalent to show that $f(Z^\circ) \subseteq X^\circ$. Let $\hat{z} \in Z^\circ$ and $x = f(\hat{z})$. We shall show that $\partial \Sigma_x = \emptyset$. By the gluing of spaces of directions, $\Sigma_{f^{-1}(x)}$ and Σ_x satisfy the assumption as in Theorem 1.1. Note that $\partial \Sigma_{\hat{z}} = \emptyset$. By Theorem 1.1, Σ_x is isometric to $\Sigma_{\hat{z}}$, which has no boundary. \square

It remains to prove Theorem 1.1(iii) and (iv).

Proof of Theorem 1.1(iii). Let $\hat{z}_\beta \in Z_{\alpha_\beta}$ satisfying $f(\hat{z}_\beta) = x \in X$. Let $d_{\alpha_\beta} = \text{diam}(Z_{\alpha_\beta})$. For each $1 \leq \beta \leq m$,

$$v_0 \leq \text{vol}(Z_{\alpha_\beta}) \leq \text{vol}(\Sigma_{\hat{z}_\beta}) \cdot \int_0^{d_{\alpha_\beta}} \text{sn}_\kappa^{n-1}(t) dt \leq \text{vol}(\Sigma_{\hat{z}_\beta}) \cdot \int_0^{d_0} \text{sn}_\kappa^{n-1}(t) dt.$$

Summing up for $\beta = 1, 2, \dots, m$, we get

$$m \cdot v_0 \leq \sum_{\beta=1}^m \text{vol}(\Sigma_{\hat{z}_\beta}) \cdot \int_0^{d_0} \text{sn}_\kappa^{n-1}(t) dt.$$

By Theorem 1.4, we have

$$\sum_{\beta=1}^m \text{vol}(\Sigma_{\hat{z}_\beta}) = \text{vol}(\Sigma_x) \leq \text{vol}(\mathbb{S}_1^{n-1}).$$

Thus

$$m \cdot v_0 \leq \text{vol}(\mathbb{S}_1^{n-1}) \cdot \int_0^{d_0} \text{sn}_\kappa^{n-1}(t) dt = \text{vol}(B_{d_0}(\mathbb{S}_\kappa^n))$$

\square

Proof of Theorem 1.1(iv). Due to Lemma 3.8(ii), it's sufficient to show that

$$\dim_H \left(\bigcup_{m=3}^{m_0} G_X^m \right) \leq n - 2.$$

By Theorem 1.1(ii) and because $m_0 < \infty$, we get

$$\dim_H \left(\bigcup_{m=3}^{m_0} G_Z^m \right) \leq n - 2.$$

We claim that there is $\delta > 0$ small such that for any $x \in G_X^m$, $m \geq 3$, there is $\hat{z} \in Z \setminus Z^{(n-1, \delta)}$ with $f(\hat{z}) = x$. Then because f is 1-Lipschitz and by Theorem 2.4, we have

$$\dim_H \left(\bigcup_{m=3}^{m_0} G_X^m \right) \leq \dim_H \left(Z \setminus Z^{(n-1, \delta)} \right) \leq n - 2.$$

If the claim is not true, then $f^{-1}(x) = \{\hat{z}_1, \hat{z}_2, \dots, \hat{z}_m\} \subset Z^{(n-1, \delta)}$. By Theorem 2.1, either $\hat{z}_i \in Z^{\tau(\delta)}$ or $z_i \in \partial Z$. In both cases, by Theorem 2.2 and 2.3, we have $\text{vol}(\Sigma_{\hat{z}_\beta}) \geq \frac{1}{2} \text{vol}(\mathbb{S}_1^{n-1}) - \tau(\delta)$ for each $1 \leq \beta \leq m$. Thus

$$\begin{aligned} \text{vol}(\mathbb{S}_1^{n-1}) &\geq \text{vol}(\Sigma_x) = \sum_{\beta=1}^m \text{vol}(\Sigma_{\hat{z}_\beta}) \\ &\geq \sum_{\beta=1}^m \left(\frac{1}{2} \text{vol}(\mathbb{S}_1^{n-1}) - \tau(\delta) \right) \geq \frac{m}{2} \text{vol}(\mathbb{S}_1^{n-1}) - m_0 \tau(\delta). \end{aligned}$$

This is impossible for $m \geq 3$ and $\delta > 0$ small. \square

5. APPLICATIONS

We can use the Shrinking rigidity Theorem to study the shrinking of spaces of directions in Alexandrov spaces. Let $X \in \text{Alex}^n(\kappa)$. When the points converge in X , the space of directions of the limit point is known to be “smaller” than the Gromov-Hausdorff limit of the spaces of directions of the sequence, i.e., we have $\liminf_{i \rightarrow \infty} \Sigma_{p_i} \geq \Sigma_p$ in the following sense.

Theorem 5.1 ([BGP] 7.14). *Let $X \in \text{Alex}^n(\kappa)$ and $p_i \rightarrow p$ be a sequence of convergent point on X . Then for any Gromov-Hausdorff convergence subsequence $\Sigma_{p_i} \xrightarrow{d_{GH}} \Sigma$, there is a 1-Lipschitz onto map $f : \Sigma \rightarrow \Sigma_p$.*

A natural question to ask is, when do we have $\lim_{i \rightarrow \infty} \Sigma_{p_i} = \Sigma_p$? Petrunin showed that if the points are interior points of a geodesic, then the spaces of directions are isometric to each other. Consequently, $\lim_{i \rightarrow \infty} \Sigma_{p_i} = \Sigma_p$ in Theorem 5.1 for points converging within the interior of a fixed geodesic.

Theorem 5.2 ([Pe2]). *Let $X \in \text{Alex}^n(\kappa)$. Then for any $x, y \in]pq[_X$, Σ_x is isometric to Σ_y .*

We find a volume condition which can determine that $\lim_{i \rightarrow \infty} \Sigma_{p_i} = \Sigma_p$. An application of this theorem is to prove the stability for relatively almost maximum volume (Theorem 5.5).

Theorem 5.3. (*Shrinking rigidity of space of directions*) *Let $X_i \in \text{Alex}^n(\kappa)$ with $(X_i, p_i) \xrightarrow{d_{GH}} (X, p)$. If $\lim_{i \rightarrow \infty} \text{vol}(\Sigma_{p_i}) = \text{vol}(\Sigma_p)$, then $\lim_{i \rightarrow \infty} \Sigma_{p_i} = \Sigma_p$.*

Proof. Not losing generality, assume that Σ_{p_i} converges and Σ is the limit. We shall first find a 1-Lipschitz onto map $f : \Sigma \rightarrow \Sigma_p$. Since it preserves the volume, by Corollary 1.3, it's sufficient to show that $f(\partial\Sigma) \subseteq \partial\Sigma_p$ if $\partial\Sigma \neq \emptyset$.

For any $\xi \in \Sigma$, let $\xi_i \in \Sigma_{p_i}$ such that $\lim_{i \rightarrow \infty} \xi_i = \xi$. Let γ_i be a quasi-geodesic in X_i with $\gamma_i(0) = p_i$ and $\gamma_i^+(0) = \xi_i$. Passing to a convergent subsequence, $\gamma = \lim_{k \rightarrow \infty} \gamma_{i_k}$ is a quasi-geodesic in X with $\gamma(0) = p$. We define

$$f : \Sigma \rightarrow \Sigma_p, \quad \xi \rightarrow \gamma^+(0).$$

By the semi-continuity of convergent angles (Theorem 5.1), f is 1-Lipschitz. By the same reason, f is independent of the selection of the sequence γ_{i_k} and ξ_i . For any $q \in X$, there are $q_i \in X_i$

such that $\lim_{i \rightarrow \infty} q_i = q$. Select geodesics $[p_i q_i]_{X_i}$ such that $\lim_{i \rightarrow \infty} [p_i q_i]_{X_i} = [pq]_X$. Passing to a subsequence, let $\xi = \lim_{k \rightarrow \infty} \uparrow_{p_{i_k}}^{q_{i_k}} \in \Sigma$. Then by our definition, $f(\xi) = \eta$. Furthermore, f is onto since $\{\uparrow_p^q : q \in X\}$ is dense in Σ_p and Σ, Σ_p are both compact.

If $\partial\Sigma \neq \emptyset$, we check that $f(\partial\Sigma) \subseteq \partial\Sigma_p$. Not losing generality, assume $\partial\Sigma_{p_i} \neq \emptyset$ for all i . Let $\xi \in \partial\Sigma$ and $\xi_i \in \partial\Sigma_{p_i}$ with $\lim_{i \rightarrow \infty} \xi_i = \xi$. Let γ_i be quasi-geodesics in X_i with $\gamma_i(0) = p_i$ and $\gamma_i^+(0) = \xi_i$. Because ∂X_i is an extremal subset, $\gamma_i \subset \partial X_i$, and thus $\gamma = \lim_{k \rightarrow \infty} \gamma_i \subset \partial X$. Therefore, $f(\xi) = \gamma^+(0) \in \partial\Sigma_p$. \square

Using the Shrinking rigidity Theorem and the above theorem, we are able to classify the Alexandrov spaces which achieve/almost achieve their relatively maximum volume. Let $C_\kappa(\Sigma_p)$ be the κ -cone (see [BGP] §4) and $C_\kappa^r(\Sigma_p)$ be the metric r -ball in $C_\kappa(\Sigma_p)$ centered at the cone vertex O . Let $\Sigma \times \{R\} = \{\tilde{q} \in \bar{C}_\kappa^R(\Sigma) : |O\tilde{q}| = R\}$ denote the “bottom” of $\bar{C}_\kappa^R(\Sigma)$, where \bar{A} denotes the closure of a subset A .

Theorem 5.4. (*Relatively Maximum Volume*) *Let $p \in X \in \text{Alex}^n(\kappa)$. For any $0 < r < R$, if the equality in the Bishop-Gromov relative volume comparison*

$$\frac{\text{vol}(B_R(p))}{\text{vol}(B_r(p))} \leq \frac{\text{vol}(B_R(\mathbb{S}_\kappa^n))}{\text{vol}(B_r(\mathbb{S}_\kappa^n))}$$

holds, then the metric ball $B_R(p)$ is isometric to $C_\kappa^R(\Sigma_p)$ in terms of their intrinsic metrics. If $X = \bar{B}_R(p)$, then

- (i) $R \leq \frac{\pi}{2\sqrt{\kappa}}$ or $R = \frac{\pi}{\sqrt{\kappa}}$ for $\kappa > 0$;
- (ii) X is isometric to a self-glued space $\bar{C}_\kappa^R(\Sigma_p)/(x \sim \phi(x))$, where $\phi : \Sigma_p \times \{R\} \rightarrow \Sigma_p \times \{R\}$ is an isometric involution;
- (iii) if X is a topological manifold, then X is homeomorphic to \mathbb{S}_1^n or $\mathbb{R}P^n$.

The above theorem was proved in [LR2] using a different technique, which relies on the specialty of cone structure in both parts of the open ball isometry and the isometric involutorial gluing. Here we will give a direct proof using Theorem 1.1. The first work in this kind in Riemannian geometry was by Grove and Petersen ([GP]), where X is assumed to be a limit of Riemannian manifolds with $\text{vol}(X) = \text{vol}(B_R(\mathbb{S}_\kappa^n))$ and the conclusion is somewhat stronger. The case assuming $X \in \text{Alex}^n(\kappa)$ with $\text{vol}(X) = \text{vol}(B_R(\mathbb{S}_\kappa^n))$ was discussed in [Sh].

Proof of Theorem 5.4. We first prove (ii). By Lemma 4.3 in [LR2], we see that if the equality holds, then $\text{vol}(B_R(p)) = \text{vol}(C_\kappa^R(\Sigma_p))$. For each $R > 0$, the gradient exponential map ([Pe3]) $g \exp_p : C_\kappa^R(\Sigma_p) \rightarrow B_R(p)$ is shrinking. Note that the proof of Lemma 1.6 relies only on the local structure of Alexandrov spaces. Thus $g \exp_p|_{C_\kappa^R(\Sigma_p)} = g \exp_p|_{C_\kappa^R(\Sigma_p)^\circ}$ is an isometry. It's clear that $g \exp_p(C_\kappa^R(\partial\Sigma_p)) \subseteq \partial B_R(p)$. Therefore $g \exp_p|_{C_\kappa^R(\Sigma_p)}$ is an isometry.

If $X = \bar{B}_R(p)$, by the above and Theorem 1.1, X is a space produced from $\bar{C}_\kappa^R(\Sigma_p)$ via a self-gluing along $\Sigma \times \{R\}$. By the same argument in [LR2] (Lemma 2.6), we see that for any $q \in \Sigma \times \{R\}$ with $\{\hat{q}_1, \hat{q}_2\} \subseteq g \exp_p^{-1}(q)$, $g \exp_p\left([O\hat{q}_1]_{\bar{C}_\kappa^R(\Sigma_p)}\right) \cup g \exp_p\left([O\hat{q}_2]_{\bar{C}_\kappa^R(\Sigma_p)}\right)$ forms a local geodesic at q . Thus $G_Z^m = \emptyset$ for $m \geq 3$, then the isometric involution follows by Theorem 1.1.

(i) follows by (ii) since if $R > \frac{\pi}{2\sqrt{\kappa}}$, then X is homeomorphic to the suspension $C_1(\Sigma_p)$. However, we have showed that the maximum gluing number $m_0 \leq 2$. Assertion (iii) follows by the same as argument in [LR2]. \square

Using Perel'man's Stability Theorem, Theorem 5.4 and the Shrinking rigidity Theorem for spaces of directions (Theorem 5.3), we get the following stability theorem, which generalizes the result in [LR2] without assuming that X is a topological manifold.

Theorem 5.5. (*Stability of Relatively Maximum Volume*) *Let $p \in X \in \text{Alex}^n(\kappa)$ with $X = \bar{B}_R(p)$. There is a constant*

$$\epsilon = \epsilon(\Sigma_p, n, \kappa, R) > 0$$

such that if $\text{vol}(X) > \text{vol}(\bar{C}_\kappa^R(\Sigma_p)) - \epsilon$, then X is homeomorphic to a self-glued space $\bar{C}_\kappa^R(\Sigma_p)/(x \sim \phi(x))$, where $\phi : \Sigma_p \times \{R\} \rightarrow \Sigma_p \times \{R\}$ is an isometric involution. In particular, if X is a topological manifold, then X is homeomorphic to \mathbb{S}_1^n or $\mathbb{R}P^n$.

Proof. Let $(X_i, p_i) \in \text{Alex}^n(\kappa)$ be a Gromov-Hausdorff convergent sequence with $\Sigma_{p_i} = \Sigma_p$, $X_i = \bar{B}_R(p_i)$ for all i , and $\lim_{i \rightarrow \infty} \text{vol}(X_i) = \text{vol}(\bar{C}_\kappa^R(\Sigma_p))$. Let (X, \bar{p}) be the limit space of (X_i, p_i) . Then $X = \bar{B}_R(\bar{p})$, $\text{vol}(X) = \text{vol}(\bar{C}_\kappa^R(\Sigma_p))$ and there is a shrinking map $f : \Sigma_p \rightarrow \Sigma_{\bar{p}}$. Consequently,

$$(5.1) \quad \text{vol}(\bar{C}_\kappa^R(\Sigma_{\bar{p}})) \leq \text{vol}(\bar{C}_\kappa^R(\Sigma_p)) = \text{vol}(X) \leq \text{vol}(\bar{C}_\kappa^R(\Sigma_{\bar{p}})).$$

Thus $\text{vol}(X) = \text{vol}(\bar{C}_\kappa^R(\Sigma_{\bar{p}}))$. By Theorem 5.4, (X, \bar{p}) is isometric to a self-glued space $\bar{C}_\kappa^R(\Sigma_{\bar{p}})/(x \sim \phi(x))$, where $\phi : \Sigma_{\bar{p}} \times \{R\} \rightarrow \Sigma_{\bar{p}} \times \{R\}$ is an isometric involution. By (5.1) again, we see that $\text{vol}(\Sigma_{\bar{p}}) = \text{vol}(\Sigma_p)$. Thus $\Sigma_{\bar{p}}$ is isometric to Σ_p by Theorem 5.3. Then the theorem follows by Perel'man's Stability Theorem. \square

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